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An algebraic generalization of Frege structures – binding algebras

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Abstract

This paper introduces a kind of second-order algebras, called *binding algebras*, which is an algebraic generalization of Aczel's (1980) Frege structure. It investigates the universal property of binding algebras by following Birkhoff's (1935) method. An equational logic $\vdash_{\mathrm{eBA}}$ is introduced. We obtain its admissible completeness. The relationship between completeness and the admissible completeness is very close and will be discussed. © 1999 — Elsevier Science B.V. All rights reserved

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1. Introduction and overview

Aczel introduced the concept of "*Frege structures*" with the intention of giving a coherent context for the rigorous development of Frege's logical notion of sets and an explanation of Russell's paradox [1]. It is a significant contribution to the theory of rules and proofs in constructive mathematics. For some applications of Frege structures, see [3, 9, 14, 15].

In a framework for binding operators [29], we introduce an algebraic generalization of Frege structures, i.e. *binding algebras*. For such a generalization, we take arbitrary finite numbers of *bindings* as primitives and extend the usual first-order algebra to a second-order algebra, called an *extensional binding algebra* (eBA). This eBA, **B**, has a signature of $\Sigma_{Nat^* \times Nat}$, where *Nat* is the set of natural numbers, since we only consider the single-sorted case. The operators σ's in signature $\Sigma_{Nat^* \times Nat}$ are of second-order, and they are called *binding operators* (BOs). For example, the existential quantifier $\exists$ of first-order logic, the lambda-abstraction operator λ and the application operator **app**

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of Lambda calculus are BOs with their signatures of $\exists, \lambda \in \Sigma_{\langle 1,0 \rangle}$ and $\mathbf{app} \in \Sigma_{\langle \varepsilon,2 \rangle}$ where ε means the empty list. The carrier of an eBA, $\mathbf{B}$, is a pair $\langle B, \{\mathscr{F}_m \mid m \in Nat\}\rangle$ where B is a set of ordinary objects and $\mathscr{F}_m$ is some part of the function space $B^m \to B$. These carriers are *explicitly closed* in Aczel's sense [1], i.e. they are closed under constant functions, projection functions and function compositions. The interpretations $\mathscr{B}_\sigma$'s of BOs σ's over an eBA, $\mathbf{B}$, are second-order functionals which satisfy the conditions of Aczel's $\mathscr{F}$*-functional* [1] (this is called the *uniformity* in this paper). It turns out that an eBA is a generalization of the following:

(i) first-order algebras,
(ii) Kechris and Moschovakis' suitable class of functionals [17],
(iii) Volken's λ-family in [2, p. 127],
(iv) Scott–Plotkin's $P\omega$-model of lambda calculus [26],
(v) Girard's qualitative domain of F system [10].

Naturally, following Birkhoff [4], we would like to characterize eBAs equationally. Kechris and Moschovakis' Enumeration Theorem [17] suggests that such an algebraic characterization might be possible. That is, we expect that a certain Birkhoff-like theorem which establishes that the following four statements are equivalent:

(a) $\mathbf{B} \models_{\mathrm{eBA}} p \simeq q$, where p and q are (binding) terms (i.e. including ordinary terms and function terms);
(b) $\zeta(\bullet_{[p]}) = \zeta(\bullet_{[q]})$ for every $\zeta : \mathbf{T} \to \mathbf{B}$, where $\mathbf{T}$ is the term eBA, ζ is a (binding) homomorphism, and $\bullet_{[p]}$ and $\bullet_{[q]}$ mean the elements in $\mathbf{T}$ designated by binding terms p and q respectively (see the comments after Definition 3.1.4);
(c) $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \bigcap_{\zeta:\mathbf{T}\to\mathbf{B}} Ker(\zeta)$, where $Ker(\zeta)$ is the (binding) kernel of (binding) homomorphism ζ, and $\bullet_{[p]}$ and $\bullet_{[q]}$ mean the elements in $\mathbf{T}$ designated by binding terms p and q, respectively;
(d) $\mathbf{T}/\theta \models_{\mathrm{eBA}} p \simeq q$, where $\mathbf{T}/\theta$ is a quotient eBA and θ is a binding congruence such that $\theta = \bigcap_{\zeta:\mathbf{T}\to\mathbf{B}} Ker(\zeta)$.

Unfortunately, eBAs and the usual (binding) satisfaction $\models_{\mathrm{eBA}}$ of binding equations (in Birkhoff's approach) do not work, i.e. (b) and (c) above are not equivalent. We, therefore, seek a remedy.

To this end, we find a condition on the satisfaction $\models_{\mathrm{eBA}}$ which makes the Birkhoff's approach work. This condition is necessary and sufficient for equivalence between (b) and (c) to hold, and we call it an *admissible* condition. That is, let $Mod(\Gamma)$ be the class of eBAs satisfying Γ (Γ is a set of binding equations $p \simeq q$), the equivalence between (1) and (2) below is valid iff we, in general, relax the satisfaction $\models_{\mathrm{eBA}}$ to the admissible satisfaction $\dot{\models}_{\mathrm{eBA}}$ (where the dot $\dot{}$ on top of $\models_{\mathrm{eBA}}$ means *admissibility*) and replace $Mod(\Gamma)$ by a larger class $Adm(\Gamma)$, where $\mathbf{B} \in Adm(\Gamma)$ if $\mathbf{B} \dot{\models}_{\mathrm{eBA}} \Gamma$:

(1) $Mod(\Gamma) \models_{\mathrm{eBA}} p \simeq q$.
(2) $\mathbf{T}/\vartheta \models_{\mathrm{eBA}} p \simeq q$ where ϑ is $\bigcap_{\mathbf{B}\in Mod(\Gamma)} Ker(\mathbf{B})$ and $Ker(\mathbf{B})$ is $\bigcap_{\zeta:\mathbf{T}\to\mathbf{B}} Ker(\zeta)$.

This admissible condition turns out to be weaker than Plotkin's logical relations [23, 27] in the sense that "logical" implies "admissible". An equational logic $\dot{\vdash}_{\mathrm{eBA}}$ for binding equations is obtained, which is admissibly sound and complete (i.e. *admissible equational logic*). This admissible equational logic $\vdash_{\mathrm{eBA}}$ is defined in the judgement form of

$$\Gamma \vdash_{\mathrm{eBA}} p \simeq q \text{ or simply } \Gamma \vdash p \simeq q.$$

It has *identity* rule $\{p \simeq q\} \vdash p \simeq q$, *reflexivity* rule $\vdash p \simeq p$, *symmetry* rule:

$$\frac{\Gamma \vdash p \simeq q}{\Gamma \vdash q \simeq p}$$

transitivity rule

$$\frac{\Gamma \vdash p \simeq q, q \simeq r}{\Gamma \vdash p \simeq r}$$

weakening rule

$$\frac{\Gamma \vdash p \simeq q}{\Gamma \cup \Gamma' \vdash p \simeq q}$$

cut rule (or *Modus Ponens*)

$$\frac{\Gamma \vdash p_1 \simeq q_1, p_2 \simeq q_2, \ldots, p_k \simeq q_k; \{p_1 \simeq q_1, p_2 \simeq q_2, \ldots, p_k \simeq q_k\} \vdash p \simeq q}{\Gamma \vdash p \simeq q}$$

with extra rules in the following:

1. (α)

   $$\vdash \langle \vec{x} : t \rangle \simeq \langle \vec{y} : t[\vec{x} := \vec{y}] \rangle$$

   where $\langle \vec{x} : t \rangle$ is a function term and distinct (ordinary) variables in list $\vec{x}$ are bound in (ordinary) term t, variables in list $\vec{y}$ do not occur in $FreeVar(t) - \{\vec{x}\}$ and $t[\vec{x} := \vec{y}]$ is the result of the usual simultaneously substituting $\vec{y}$ for $\vec{x}$ (i.e. an ordinary variable substitution).
2. (ξ)

   $$\frac{\Gamma \vdash t \simeq u}{\Gamma \vdash \langle \vec{x} : t \rangle \simeq \langle \vec{x} : u \rangle}$$

3. (ξ^{-1})

   $$\frac{\Gamma \vdash \langle \vec{y} : t \rangle \simeq \langle \vec{z} : u \rangle}{\Gamma \vdash t[\vec{y} := \vec{x}] \simeq u[\vec{z} := \vec{x}]}$$

   where ordinary variables in list $\vec{x}$ do not occur in either $FreeVar(t) - \{\vec{y}\}$ or $FreeVar(u) - \{\vec{z}\}$.
4. (b-sub)

   $$\frac{\Gamma \vdash p \simeq q}{\Gamma \vdash p[\vec{f}, \vec{x} := \vec{ft}, \vec{u}] \simeq q[\vec{f}, \vec{x} := \vec{ft}, \vec{u}]}$$

   where $\vec{f}$ is a list of distinct function variables, $\vec{x}$ is a list of distinct ordinary variables, $\vec{ft}$ is a list of function terms, and $\vec{u}$ is a list of ordinary terms.

5. (cmp-1)

$$\frac{\Gamma \vdash t_1 \simeq u_1, t_2 \simeq u_2, \ldots, t_n \simeq u_n}{\Gamma \vdash f(t_1, t_2, \ldots, t_n) = f(u_1, u_2, \ldots, u_n)}$$

where f is a function variable with arity n.

6. (cmp-2)

$$\frac{\Gamma \vdash ft_1 \simeq fu_1, ft_2 \simeq fu_2, \ldots, ft_\ell \simeq fu_\ell, t_1 \simeq u_1, t_2 \simeq u_2, \ldots, t_n \simeq u_n}{\Gamma \vdash \sigma(ft_1, ft_2, \ldots, ft_\ell, t_1, t_2, \ldots, t_n) \simeq \sigma(fu_1, fu_2, \ldots, fu_\ell, u_1, u_2, \ldots, u_n)}$$

where $\sigma \in \Sigma_{\langle \vec{m}, n\rangle}$ with $\ell = |\vec{m}|$ (the length of list $\vec{m}$), ft_i and fu_i are function terms with arity m_i (ith element of list $\vec{m}$ and $1 \leqslant i \leqslant \ell$), t_j and u_j are ordinary terms with $1 \leqslant j \leqslant n$.

The relation between the satisfaction $\models_{\mathrm{eBA}}$ and the admissible satisfaction $\dot{\models}_{\mathrm{eBA}}$ will be discussed in Section 9.3, although the relationship is not completely worked out. Some problems remain open.

For some applications of the equational logic $\vdash_{\mathrm{eBA}}$, see [24, 29, 30].

Meinke gives a systematic account of general higher-order algebras in [19]. In contrast to ours, his approach is to use the non-standard method introduced by Henkin. The inter-relationship between his results and ours certainly merits a futher investigation. He also presents a reason for the importance of second-order algebras in [20]. Möller treats higher-order algebras by using ordered approach [21]. Others such as Poigné, Dybjer, Parsaye-Ghomi, Maibaum and Lucena also treat (general) higher-order algebras by using category theory [25, 7, 22, 18]. Our treatment of specific second-order algebras (binding algebras) in Birkhoff's approach is believed to be unique if not the best. A connection between binding algebras and either cartesian-closed categories or 2-categories is treated in [29, pp. 288–292, 54–55], respectively. This connection will not be explored here.

The paper is organized into 9 sections as follows. Section 2 introduces many basics of extensional binding alegbras (eBAs); e.g. Section 2.1 briefly introduces the syntax of binding algebras (i.e. binding terms), Section 2.2 gives the formal definitions of eBAs and the satisfaction $\models_{\mathrm{eBA}}$, Section 2.3 introduces the concepts of sub-eBAs (subalgebras), perfect sub-eBAs (subalgebras with the extensionality) and binding homomorphisms (eBHs), Section 2.4 discusses the properties of sub-eBAs and generated sub-eBAs, and Section 2.5 presents the relationships between eBHs and the generated eBAs. Section 3 introduces the term eBA **T**; e.g. Section 3.1 constructs the term eBA, and Section 3.2 discusses the generatability of the term eBA. Section 4 is devoted to the universal property of eBAs – the free eBAs. Section 5 introduces the binding congruences (eBCs) and the quotient eBAs. Section 6 introduces the admissible freeness of eBAs. Section 7 gives the definitions of the admissible binding equations and the admissible Birkhoff's theorem. Section 8 proves the admissible completeness of the equational logic $\vdash_{\mathrm{eBA}}$. Some discussions are provided in Section 9, where Section 9.1 discusses the admissible variety problem of eBAs, Section 9.2 gives the relationship between Plotkin's logical relations and the admissibility, and finally, and Section 9.3

discusses the relationship between the completeness and the admissible completeness of the equational logic $\vdash_{\text{eBA}}$. Since all proofs involved in this paper can be found in [29], almost all proofs are left out for simplicity.

2. Extensional binding algebras (eBAs)

2.1. Binding terms (BTs)

Let V be a countably infinite set of ordinary variables with a linear order; let $x, y, z \ldots$ range over V; $\vec{x}, \vec{y}, \vec{z} \ldots$ range over lists of (distinct) ordinary variables. Let $\vec{F}$ be a family of F_m (the set of function variables with arity m, where $m \geqslant 0$) and $f, \ldots$ range over $\vec{F}$. For all $m \geqslant 0$, F_m and V are all disjoint from each other.

Definition 2.1.1 (*Binding terms BT*). Suppose that Σ is the (binding) signature and BT (binding terms) is a pair of $\langle T, FT \rangle$ where T is the set of ordinary terms and FT is the family of FT_m the set of function terms with arity m ($m \in Nat$). They are generated by using the formation rules:

1. $$\frac{x \in V}{x \in T}$$

2. $$\frac{t_1, t_2, \ldots, t_n \in T}{f(t_1, t_2, \ldots, t_n) \in T} \ (f \in F_n \text{ and } n \in Nat)$$

3. $$\frac{t \in T; x_1, x_2, \ldots, x_k \in V}{\langle x_1, x_2, \ldots, x_k : t \rangle \in FT_k} \ (x_i \neq x_j \text{ if } i \neq j)$$

4. for $\langle \vec{m}, n \rangle \in Nat^* \times Nat$ and $\ell = |\vec{m}|$ (the length of list $\vec{m}$)

$$\frac{ft_1 \in FT_{m_1}, ft_2 \in FT_{m_2}, \ldots, ft_\ell \in FT_{m_\ell}; t_1, t_2, \ldots, t_n \in T}{\sigma(ft_1, ft_2, \ldots, ft_\ell, t_1, t_2, \ldots, t_n) \in T} \quad (\sigma \in \Sigma_{\langle \vec{m}, n \rangle})$$

The above definition deserves some explanation:

(1) for a conceptual reason, we keep zero (null) binding term $\langle \varepsilon : t \rangle$ different from ordinary term t;

(2) when $n = 0$, the second condition above becomes

$$\frac{f \in F_0}{f() \in T};$$

(3) when $k = 0$, the third condition becomes

$$\frac{t \in T}{\langle \varepsilon : t \rangle \in FT_0};$$

(4) when the length of list $\vec{m}$ is zero (i.e. $\ell=0$), the fourth condition becomes

$$\frac{t_1,t_2,\ldots,t_n\in T}{\sigma(t_1,t_2,\ldots,t_n)\in T}\ (\sigma\in\Sigma_{\langle\varepsilon,n\rangle})$$

(i.e. this σ is a usual first-order operator); furthermore, if also $n=0$, it will become

$$\frac{}{\sigma()\in T}\ (\sigma\in\Sigma_{\langle\varepsilon,0\rangle})$$

(i.e. this σ is the usual constant operator);

(5) when $n=0$, the fourth condition becomes

$$\frac{ft_1\in FT_{m_1},ft_2\in FT_{m_2},\ldots,ft_\ell\in FT_{m_\ell}}{\sigma(ft_1,ft_2,\ldots,ft_\ell)\in T}\ (\sigma\in\Sigma_{\langle\vec{m},0\rangle}).$$

Also, we obviously have $F_m\cap FT_m=\emptyset$ ($m\geqslant 0$) and $V\subseteq T$.

Later, for convenience, we introduce some abbreviation as follows:

(a) $f(t_1,t_2,\ldots,t_n)$ is often abbreviated as $f(\vec{t})$ such that $n=|\vec{t}|$ and t_j is the jth element of list $\vec{t}$;

(b) $\langle x_1,x_2,\ldots,x_k:t\rangle$ is sometimes shortened as $\langle\vec{x}:t\rangle$ where $k=|\vec{x}|$;

(c) $\sigma(ft_1,ft_2,\ldots,ft_\ell,t_1,t_2,\ldots,t_n)$ is usually written as $\sigma(\vec{ft},\vec{t})$ where $\ell=|\vec{ft}|$ and $n=|\vec{t}|$.

2.2. *eBAs and their satisfaction* $\models_{\mathrm{eBA}}$

Let B be a set of ordinary objects, and let $\mathscr{F}_n$ be a subset of function space $B^n\to B$ for $n\geqslant 0$.

Definition 2.2.1 (*Explicitly closedness*). A family $\langle B,\{\mathscr{F}_n\mid n\in Nat\}\rangle$ is called *explicitly closed* iff

1. (constant) for each $n\geqslant 0$ and $b\in B$, there is a unique function $C_{n,b}\in\mathscr{F}_n$ such that $C_{n,b}(\vec{a})=b$ for all $\vec{a}\in B^n$;
2. (projection) for each $n>0$ and $1\leqslant i\leqslant n$, there is a unique function, named $\pi_{n,i}\in\mathscr{F}_n$, such that $\pi_{n,i}(\vec{b})=b_i$ for all $\vec{b}\in B^n$ (where b_i is the ith element of list $\vec{b}$);
3. (composition) for $n>0$ and $k\geqslant 0$, given $g\in\mathscr{F}_n$ and $g_i\in\mathscr{F}_k$ ($1\leqslant i\leqslant n$), there is a unique function $h\in\mathscr{F}_k$ such that $h(\vec{b})=g(g_1(\vec{b}),g_2(\vec{b}),\ldots,g_n(\vec{b}))$ or $h=g\circ\langle g_1,g_2,\ldots,g_n\rangle$; sometimes it is further abbreviated as $g(\vec{g})$.

Informally, the explicit closedness of a family means that the family is closed under constant functions, projection functions and function compositions. Such a closure is not necessarily preserved by any map over the family. We, therefore, introduce the concept of uniformity over the family.

Definition 2.2.2 (*Uniformity*). For $\sigma\in\Sigma_{\langle\vec{m},n\rangle}$ ($\ell=|\vec{m}|$), an interpretation of it is, in general, a second-order functional $\mathscr{B}_\sigma:\mathscr{F}_{m_1}\times\mathscr{F}_{m_2}\times\cdots\times\mathscr{F}_{m_\ell}\times B^n\to B$. This functional $\mathscr{B}_\sigma$ is *uniform* over an explicitly closed family $\langle B,\{\mathscr{F}_n\mid n\in Nat\}\rangle$ iff for any $k\geqslant 0$, given $g_i\in\mathscr{F}_{k+m_i}$ ($1\leqslant i\leqslant\ell$) and $g_{\ell+j}\in\mathscr{F}_k$ ($1\leqslant j\leqslant n$), the function h is in $\mathscr{F}_k$ where

$h(\vec{b}) = \mathscr{B}_\sigma(\vec{h}, \vec{c})$ for all $\vec{b} \in B^k$, where $h_i(\vec{a}) = g_i(\vec{b}, \vec{a})$ for all $\vec{a} \in B^{m_i}$ $(1 \leqslant i \leqslant \ell)$ and $c_j = g_{\ell+j}(\vec{b})$ $(1 \leqslant j \leqslant n)$.

Note that h_i (the ith element of list $\vec{h}$) is in $\mathscr{F}_{m_i}$, since

$$h_i = g_i \circ \langle C_{m_i,b_1}, C_{m_i,b_2}, \ldots, C_{m_i,b_k}, \pi_{m_i,1}, \pi_{m_i,2}, \ldots, \pi_{m_i,m_i} \rangle$$

for $1 \leqslant i \leqslant \ell$ where $b_j \in B$ and $1 \leqslant j \leqslant k = |\vec{b}|$.

Now, we are able to define an extensional binding algebra (eBA) as following.

Definition 2.2.3 (*eBA*). An extensional binding algebra (or eBA) $\mathbf{B}$ is a pair of $\langle \mathscr{F}^{\mathbf{B}}, \mathscr{B} \rangle$ such that
1. $\mathscr{F}^{\mathbf{B}} = \langle B, \{\mathscr{F}_n \mid n \in Nat\} \rangle$ is an explicitly closed family;
2. $\mathscr{B} = \bigcup_{\langle \vec{m}, n \rangle \in Nat^* \times Nat} \{\mathscr{B}_\sigma \mid \sigma \in \Sigma_{\langle \vec{m}, n \rangle}\}$ and for each $\sigma \in \Sigma_{\langle \vec{m}, n \rangle}$ $\mathscr{B}_\sigma$ is uniform over $\mathscr{F}^{\mathbf{B}}$.

For example, classical first-order algebras are special cases of eBAs. More specifically, let $\mathbf{A} = \langle A, \mathscr{A} \rangle$ be a classical first-order algebra with signature $\Sigma' = \bigcup_{n \in Nat} \Sigma'_n$ where $\sigma \in \Sigma'_n$ means that σ is a first-order operation with arity n. Let $\mathbf{B} = \langle \mathscr{F}^{\mathbf{B}}, \mathscr{B} \rangle$ be an eBA with signature $\Sigma = \bigcup_{\langle \vec{m}, n \rangle \in Nat^* \times Nat} \Sigma_{\langle \vec{m}, n \rangle}$ where (a) the signature $\Sigma_{\langle \varepsilon, n \rangle} = \Sigma'_n$ and $\Sigma_{\langle \vec{m}, n \rangle} = \emptyset$ if $\vec{m} \neq \varepsilon$, (b) carrier $B = A$ and function carrier $\mathscr{F}_n$ is the full function space from carrier A^n to A with $n \in Nat$, (c) $\mathscr{B}_\sigma = \mathscr{A}_\sigma$ for $\sigma \in \Sigma'$. Obviously, $\mathscr{F}^{\mathbf{B}}$ is explicitly closed and $\mathscr{B}_\sigma$ is uniform over full function spaces in $\mathscr{F}^{\mathbf{B}}$.

Compared to Kechris and Moschovakis' suitable classes of functionals in [17], an eBA is said to be a generalized case of them in the following sense: (a) the first condition of eBAs corresponds to the composition of "addition of variables" (closed under constants), "substitution of projections" (closed under projections) and "compositions" (closed under function compositions); (b) the second condition of eBAs corresponds to "functional substitutions"; (c) the other conditions for a suitable class of functionals are actually limited to the standard model of natural numbers. Also, their concept of *concentration of a suitable class of functionals* is similar to the explicit closedness.

Another example of eBAs is Scott–Plotkin's $P\omega$-model of lambda calculus [26]. To see this clearer, let B be $P\omega$ and $\mathscr{F}_n$ be the continuous function space from n product of $P\omega$ to $P\omega$. Apparently, this $\langle B, \{\mathscr{F}_n \mid n \in Nat\} \rangle$ is explicitly closed. Then given $k \in Nat$ we have that $[\![\lambda]\!] \circ \langle curry_{k,1}(g), h \rangle$ and $[\![\mathbf{app}]\!] \circ \langle h_1, h_2 \rangle$ are continuous for each $g \in \mathscr{F}_{k+1}$ and every $h, h_1, h_2 \in \mathscr{F}_k$. This implies that $[\![\lambda]\!]$ and $[\![\mathbf{app}]\!]$ are uniform over $\langle B, \{\mathscr{F}_n \mid n \in Nat\} \rangle$.

A fourth example of eBAs is Girard's qualitative domains in F system [10]. More specifically, let B be a qualitative domain and $\mathscr{F}_n$ be the stable function space from B^n to B. Obviously, this $\mathscr{F}^{\mathbf{B}}$ is explicitly closed. Furthermore, Theorem 1.11(i) and (ii) in [10] guarantee that $[\![\lambda]\!]$ (i.e. (i)) and $[\![\mathbf{app}]\!]$ (i.e. (ii)) are uniform over the stable function spaces in $\mathscr{F}^{\mathbf{B}}$.

Volken's λ-family is a special case of an eBA, see [29, pp. 286–288] for details.

Let $\mathbf{B}$ be an eBA, a *valuation* $\vec{\rho}$ on $\mathbf{B}$ is a family of functions ρ (from V to B) and ρ_k (from F_k to $\mathscr{F}_k$) for $k \in Nat$. Later, we will denote a valuation $\vec{\rho}$ as a pair of

$\langle \rho, \vec{\varphi} \rangle$ where $\vec{\varphi}$ is a family of φ_k ($\varphi_k = \rho_k$) for $k \in Nat$, or simply denote it as $\langle \rho, \varphi \rangle$. Sometimes, we call a valuation $\vec{\rho}$ an *environment*.

Let $\langle \rho, \varphi \rangle$ be a valuation on $\mathbf{B}$. Then for any $x \in V$ and $a \in B$, $\rho[a/x]$ can be defined by

$$\rho[a/x](y) = \begin{cases} \rho(y) & \text{if } y \neq x, \\ a & \text{if } y = x. \end{cases}$$

From this, we have (1) that $\rho[a/x][b/x] = \rho[b/x]$ for all $a, b \in B$ and (2) that $\rho[a/x][b/y] = \rho[b/y][a/x]$ if $x \neq y$, for all $a, b \in B$. Therefore, we can let $\rho[\vec{a}/\vec{x}]$ be $\rho[a_1/x_1][a_2/x_2] \ldots [a_k/x_k]$ where $|\vec{a}| = |\vec{x}| = k$, since the result does not depend on the order of the appearences of (ordinary) variables.

Definition 2.2.4 (*Interpretation in an eBA*). Let $\mathbf{B}$ be an eBA, and $\langle \rho, \varphi \rangle$ be a valuation of ordinary variables V and function variables in $\vec{F}$ on $\mathbf{B}$. An interpretation $\mathscr{B}[\![\bullet]\!](\rho, \varphi)$ of binding terms $\bullet$ in BT over $\mathbf{B}$ under the environment $\langle \rho, \varphi \rangle$ is defined inductively by

1. $\mathscr{B}[\![x]\!](\rho, \varphi) =_{\mathrm{df}} \rho(x)$ for $x \in V$;
2. $\mathscr{B}[\![f(\vec{t})]\!](\rho, \varphi) =_{\mathrm{df}} \varphi(f)(\mathscr{B}[\![\vec{t}]\!](\rho, \varphi))$ for $f \in F_n$ and $t_j \in T$ $(1 \leqslant j \leqslant n)$, where $|\vec{t}| = n$ and $\mathscr{B}[\![\vec{t}]\!](\rho, \varphi)$ is the abbreviation of the list $\mathscr{B}[\![t_1]\!](\rho, \varphi)$, $\mathscr{B}[\![t_2]\!](\rho, \varphi), \ldots, \mathscr{B}[\![t_n]\!](\rho, \varphi)$;
3. $\mathscr{B}[\![\sigma(\vec{ft}, \vec{t})]\!](\rho, \varphi) =_{\mathrm{df}} \mathscr{B}_\sigma(\mathscr{B}[\![\vec{ft}]\!](\rho, \varphi), \mathscr{B}[\![\vec{t}]\!](\rho, \varphi))$ for $\sigma \in \Sigma_{\langle \vec{m}, n \rangle}$, $ft_i \in FT_{m_i}$ $(1 \leqslant i \leqslant \ell)$ and $t_j \in T$ $(1 \leqslant j \leqslant n)$, where $|\vec{ft}| = |\vec{m}| = \ell$ and $\mathscr{B}[\![\vec{ft}]\!](\rho, \varphi)$ is the abbreviation of the list $\mathscr{B}[\![ft_1]\!](\rho, \varphi)$, $\mathscr{B}[\![ft_2]\!](\rho, \varphi), \ldots,$ $\mathscr{B}[\![ft_\ell]\!](\rho, \varphi)$;
4. $\mathscr{B}[\![\langle \vec{x} : t \rangle]\!](\rho, \varphi) =_{\mathrm{df}} g$, where $g(\vec{a}) =_{\mathrm{df}} \mathscr{B}[\![t]\!](\rho[\vec{a}/\vec{x}], \varphi)$ and $|\vec{a}| = |\vec{x}|$.

In general, a binding term p "is equal to" another binding term q, written as $p \simeq q$, if and only if (iff) all evaluations of the two binding terms under every environment are the same. Formally

Definition 2.2.5 (*Satisfaction* $\models_{\mathrm{eBA}}$). $\mathbf{B} \models_{\mathrm{eBA}} p \simeq q$ iff $\mathscr{B}[\![p]\!](\rho, \varphi) = \mathscr{B}[\![q]\!](\rho, \varphi)$ for every environment $\langle \rho, \varphi \rangle$ over an eBA $\mathbf{B}$.

Note that the admissible satisfaction $\dot{\models}_{\mathrm{eBA}}$ will be defined in Definition 7.1.

2.3. Sub-eBAs, perfect sub-eBAs and binding homomorphisms (eBHs)

As we know, first-order subalgebras have the following two properties:
1. each subalgebra is closed under functionality (or compositionality), and
2. each subalgebra is an algebra if it is restricted to its basis.

Note that the above 1 means that a function applying to elements in domains of a subalgebra results in a element still being in the domain of the subalgebra.

However, an sub-extensional binding algebra (or sub-eBA) only possesses the first property above not necessarily the second property. The loss of possession of the second property comes from the fact that function spaces are available in eBAs but not available in first-order algebras. Because of this loss, it is our intention to introduce a concept of "*perfect*" sub-eBAs alongside with sub-eBAs. This "perfect" sub-eBA is a sub-eBA with the second property. Also, this "perfectness" turns out to be very important to the admissibility in Section 6.

Definition 2.3.1 (*sub-eBA*). Let $\mathbf{B}$ be an eBA. $\mathbf{B}' = \langle \mathscr{F}^{\mathbf{B}'}, \mathscr{B}' \rangle$ is a sub-extensional binding algebra (or a sub-eBA) iff
1. $\mathscr{F}^{\mathbf{B}'} = \langle B', \{\mathscr{F}_n' \mid n \in Nat\} \rangle$ such that $B' \subseteq B$ and $\mathscr{F}_n' \subseteq \mathscr{F}_n$ for $n \in Nat$,
2. $\mathscr{F}^{\mathbf{B}'}$ is explicitly closed, and
3. for each $\sigma \in \Sigma_{\langle \vec{m}, n \rangle}$, $\mathscr{B}_\sigma' = \mathscr{B}_\sigma$ and $\mathscr{B}_\sigma'$ is uniform over $\mathscr{F}^{\mathbf{B}'}$.

Sometimes, we use $\mathbf{B}' \prec \mathbf{B}$ to mean that $\mathbf{B}'$ is a sub-eBA of $\mathbf{B}$.

Definition 2.3.2 (*perfect sub-eBA*). Let $\mathbf{B}'$ be a sub-eBA of an eBA $\mathbf{B}$, i.e. $\mathbf{B}' \prec \mathbf{B}$. $\mathbf{B}'$ is said to be a *perfect* sub-eBA of the eBA $\mathbf{B}$, written as $\mathbf{B}' \preccurlyeq \mathbf{B}$, iff $\mathbf{B}'$ has the extensionality on its basis B', i.e. for $n \geqslant 0$ and given $g, h \in \mathscr{F}_n'$, $g(\vec{a}) = h(\vec{a})$ for all $a_j \in B'$ implies $g = h$, where a_j is the jth element in the list $\vec{a}$.

It is easy to verify that a perfect sub-eBA is an eBA if we restrict it to its basis. Next, we are to introduce a concept of the (extensional) binding homomorphisms (eBHs). We understand that a (first-order) homomorphism is basically a function which preserves functionality (or compositionality). However, for binding homomorphisms, some minimal requirements are needed besides functionality. Informally, these requirements can be expressed by certain preservations, i.e. preservations of certain primitive functions like constant functions, projection functions and compositions of functions.

Definition 2.3.3 (*Binding homomorphisms – eBHs*). A (extensional) binding homomorphism (eBH) ζ from an eBA $\mathbf{B}$ to another eBA $\mathbf{B}'$, written as $\zeta : \mathbf{B} \to \mathbf{B}'$, is a family of both a function from B to B' and functionals from $\mathscr{F}_n$ to $\mathscr{F}_n'$ for each $n \in Nat$ such that
1. (functionality) $\zeta(g(\vec{b})) = \zeta(g)(\vec{a})$, where $g \in \mathscr{F}_k$ $(k \geqslant 0)$ and $a_j = \zeta(b_j)$ $(1 \leqslant j \leqslant k)$;
2. (constant) $\zeta(C_{n,b}) = C_{n,a}$, where $b \in B$, $a = \zeta(b)$ and $n \in Nat$;
3. (projection) $\zeta(\pi_{n,i}) = \pi_{n,i}'$, where $n > 0$ and $1 \leqslant i \leqslant n$;
4. (composition) $\zeta(g \circ \langle \vec{h} \rangle) = \zeta(g) \circ \langle \zeta(\vec{h}) \rangle$ where $\zeta(\vec{h})$ is the list of $\zeta(h_1), \zeta(h_2), \ldots, \zeta(h_n)$ $(n = |\vec{h}|)$;
5. (uniformity) $\zeta(\mathscr{B}_\sigma \circ \langle \vec{g'}, \vec{h} \rangle) = \mathscr{B}_\sigma' \circ \langle \vec{g''}, \vec{h'} \rangle$ for $\sigma \in \Sigma_{\langle \vec{m}, n \rangle}$ and each $g_i \in \mathscr{F}_{k+m_i}$ $(1 \leqslant i \leqslant \ell = |\vec{m}|)$ and $h_j \in \mathscr{F}_k$ $(1 \leqslant j \leqslant n)$, where $g_i' = curry_{k,m_i}(g_i)$, $g_i'' = curry_{k,m_i}(\zeta(g_i))$ and $h_j' = \zeta(h_j)$.

2.4. Sub-eBAs and the generated sub-eBAs

It can be verified that the intersection of two sub-eBAs is a sub-eBAs. Formally,

Lemma 2.4.1 (Intersection of two sub-eBAs). *Let* $\mathbf{A}'_1, \mathbf{A}'_2 \prec \mathbf{A}$. *Then,* $\mathbf{A}'_1 \cap \mathbf{A}'_2 \prec \mathbf{A}$, *where* $\mathbf{A}'_1 \cap \mathbf{A}'_2$ *is* $\langle B_1 \cap B_2, \mathscr{F}^{\mathbf{A}'_1} \cap \mathscr{F}^{\mathbf{A}'_2} \rangle$ *such that* $\mathscr{F}^{\mathbf{A}'_1} \cap \mathscr{F}^{\mathbf{A}'_2} = \{\mathscr{F}_m^{\mathbf{A}'_1} \cap \mathscr{F}_m^{\mathbf{A}'_2} \mid m \in Nat\}$.

Lemma 2.4.1 can be extended from two sub-eBAs to a class of sub-eBAs. Formally

Lemma 2.4.2 (Intersection of sub-eBAs). *Let* $\mathscr{K}$ *be a class of sub-eBAs of an eBA* $\mathbf{A}$. *Then* $\bigcap \mathscr{K} \prec \mathbf{A}$.

As an analogy to the usual first-order generated subalgebras, we introduce a concept of sub-eBAs generated by a sub-structure.

Definition 2.4.3 (*Generated sub-eBA*). Let $\mathbf{X}$ be a family of both X and X_k which are subsets of A and $\mathscr{F}_k$ of an eBA $\mathbf{A}$ ($k \geqslant 0$), respectively. The smallest sub-eBA containing $\mathbf{X}$, written as $[\mathbf{X}]$ or $Sub(\mathbf{X})$, is $\bigcap \mathscr{K}$ where $\mathscr{K} = \{\mathbf{A}' \mid \mathbf{A}' \prec \mathbf{A}$ and $\mathbf{X} \subseteq \mathbf{A}'\}$.

To justify the word "*generated*", we show how to construct the least sub-eBA $[\mathbf{X}]$, which contains $\mathbf{X}$, as follows.

Definition 2.4.4 (*Extension ext from a base*). The extension of $\mathbf{X}$, $\mathbf{ext}(\mathbf{X})$, is a family of both $ext(\mathbf{X})$ and $ext_k(\mathbf{X})$ ($k \in Nat$), where

1. $$ext(\mathbf{X}) =_{\mathrm{df}} X \cup \{g(\vec{a}) \mid g \in X_m \text{ for some } m \wedge a_j \in X\ (1 \leqslant j \leqslant m)\} \cup \{\sigma(\vec{g}, \vec{a}) \mid \sigma \in \Sigma_{\langle \vec{m}, n\rangle} \wedge g_i \in \mathbf{X}_{m_i}\ (1 \leqslant i \leqslant |\vec{m}\,|) \wedge a_j \in X\ (1 \leqslant j \leqslant n)\}$$

2. $$ext_k(\mathbf{X}) =_{\mathrm{df}} X_k \cup \{C_{k,a} \mid a \in X\} \cup \{\pi_{k,i} \mid 1 \leqslant i \leqslant k\} \cup \{g \circ \langle \vec{h} \rangle \mid g \in \mathbf{X}_m \wedge h_j \in X_k\ (1 \leqslant j \leqslant m) \text{ for } m > 0\} \cup \{\sigma^{\mathbf{A}} \circ \langle curry_{k,\vec{m}}(\vec{g}), \vec{h} \rangle \mid \sigma \in \Sigma_{\langle \vec{m}, n\rangle} \wedge g_i \in X_{m_i+k} \wedge h_j \in X_k\}.$$

Let $\mathbf{ext}^{\jmath+1}(\mathbf{X}) = \mathbf{ext}(\mathbf{ext}^{\jmath}(\mathbf{X}))$ ($\jmath \geqslant 0$) where $\mathbf{ext}^0(x\mathbf{X})$ is $\mathbf{X}$. In what follows, we first show that the extension grows larger and larger by repeatedly applying $\mathbf{ext}$. Then, we verify (a) that the least upper bound of such extensions forms a sub-eBA and (b) that it coincides with the generated sub-eBA.

Lemma 2.4.5 (Monotonicity of *ext*). (a) $\mathbf{ext}^j(X) \subseteq \mathbf{ext}^{j+1}(X)$, *for any* $j \geqslant 0$; (b) $\mathbf{ext}^j(X) \subseteq \mathbf{ext}^{j'}(X)$ *for any* $0 \leqslant j \leqslant j'$.

Therefore,

Lemma 2.4.6 (Boundness of *ext*). $\mathbf{ext}^j(\mathbf{X}) \subseteq \mathscr{F}^{\mathbf{A}}$ *for any* $j \geqslant 0$, *where* $\mathbf{X} \subseteq \mathbf{A}$.

Then, we confirm that the least upper bound of such extensions (repeatedly applying $\mathbf{ext}$ to $\mathbf{X}$ which is inside of an eBA $\mathbf{A}$) is an sub-eBA of the eBA $\mathbf{A}$.

Lemma 2.4.7 (Countably-limit of *ext*). (a) $\bigcup_{j\in\omega}\mathbf{ext}^j(\mathbf{X})$ *is explicitly closed*;

(b) *for any* $\sigma \in \Sigma_{\langle \vec{m},n\rangle}$, *its interpretation* $\sigma^{\mathbf{A}}$ *is uniform over the countable limit* $\bigcup_{j\in\omega}\mathbf{ext}^j(\mathbf{X})$.

Furthermore,

Theorem 2.4.8 (*ext* and sub-eBA). *The countable limit is a sub-eBA, i.e.* $\bigcup_{j\in\omega}\mathbf{ext}^j(\mathbf{X}) \prec \mathbf{A}$ *provided* $\mathbf{X} \subseteq \mathbf{A}$.

From Theorem 2.4.8, we know that a sub-eBA can be constructed from **X**. In what follows, we show that it is indeed the generated sub-eBA containing **X**.

From Lemma 2.4.6, we understand that $\mathbf{ext(X)} \subseteq \mathscr{F}^{[\mathbf{X}]}$ and $\mathbf{ext}^j(\mathbf{X}) \subseteq \mathscr{F}^{[\mathbf{X}]}$, for any $j \geqslant 0$. So, the countable limit $\bigcup_{j\in\omega}\mathbf{ext}^j(\mathbf{X}) \subseteq [\mathbf{X}]$. Subsequently, by both Theorem 2.4.8 and Definition 2.4.3, we have the following:

Theorem 2.4.9 (*ext* and the generated sub-eBA). *The countable limit is a generated sub-eBA, i.e.* $\bigcup_{j\in\omega}\mathbf{ext}^j(\mathbf{X}) = [\mathbf{X}]$, *provided* $\mathbf{X} \subseteq \mathbf{A}$.

This theorem justifies the terminology of "*generated*" sub-eBAs, and it says that [**X**] is the sub-eBA generated by **X**. Let **A** be an eBA and **X** be a sub-collections of A, then **A** is said to be the eBA *generated from* **X** if $[\mathbf{X}] = \mathbf{A}$. From this, we are led to the following: the restriction of a perfect sub-algebra to its base forms a new eBA. Formally,

Lemma 2.4.10 (Perfect sub-eBA and eBA). *Let* **A** *be an eBA, and* $\mathbf{A}'$ *be a perfect sub-eBA of the eBA* **A** (*i.e.* $\mathbf{A}' \preceq \mathbf{A}$). *Then, the result of restricting* $\mathbf{A}'$ *to its base* A', *i.e.* $\mathbf{A}'\lceil_{A'}$, *is an (new) eBA, where* $\mathscr{F}^{(\mathbf{A}'\lceil_{A'})} = (\mathscr{F}^{\mathbf{A}'})\lceil_{A'} = \{\mathscr{F}_m^{\mathbf{A}'}\lceil_{A'} \mid m \in Nat\}$.

Note that the domain A^m of a function $g : A^m \to A$ in $\mathscr{F}(A)^{\mathbf{A}'}$ is in a product of multiple copies of A for some $m \geqslant 0$; and $\mathscr{F}(A)^{\mathbf{A}'}\lceil_{A'}$ is to limit the domains of its functions (e.g. g) to the products $(A')^m$ of multiple copies of A' where $A' \subseteq A$.

The large difference between an eBA and a sub-eBA (or between a perfect sub-eBA and a sub-eBA) is the extensionality. Because of the extensionality and the presence of function carriers in eBAs, we lose some important properties. For example, see Theorem 5.16, whether can we drop off the condition "*onto*" on an ordinary eBH? This "*dropping-off*" plays a crucial role in showing that an eBA is a *free* eBA over a class $\mathscr{K}$ of eBAs. To obtain a positive answer, we consequently have to introduce the "*admissible*" concept in Section 6.

2.5. *eBHs and their uniqueness over generated eBAs*

This section is mainly to prove Theorem 2.5.9, which is important in Section 4 (related to the free eBAs). First, we show that binding homomorphims (eBHs) preserve "*explicit closedness*" (Lemma 2.5.1). The preservation of the uniformity under eBHs follows after that (Lemma 2.5.2). Therefore, an image of an eBH is a sub-eBA (Lemma 2.5.3).

Lemma 2.5.1 (Explicit-closedness preserved by eBHs). *Let* $\mathbf{A}$ *and* $\mathbf{A}'$ *be eBAs, where* $\mathbf{A}=\langle A,\{\mathscr{F}_m \mid m \in Nat\}\rangle$; ζ *be an eBH from* $\mathbf{A}$ *to* $\mathbf{A}'$. *Thus, if a sub-family* $\langle A',\{\mathscr{G}_k \mid k \in Nat\}\rangle$ *of the family* $\langle A,\{\mathscr{F}_k \mid k \in Nat\}\rangle$ *is explicitly closed, so is* $\zeta(\mathscr{G})$.

Second, eBHs preserve the "*uniformity*" over explicitly-closed families. Formally,

Lemma 2.5.2 (Uniformity preserved by eBHs). *Let* $\mathbf{A}$ *and* $\mathbf{A}'$ *be eBAs, where* $\mathbf{A}=\langle A,\{\mathscr{F}_m \mid m \in Nat\}\rangle$; ζ *be an eBH from* $\mathbf{A}$ *to* $\mathbf{A}'$; $\mathscr{G} \subseteq \mathscr{F}$ *be explicitly closed. Then, if an interpretation* $\sigma^{\mathbf{A}}$ *is uniform over* $\mathscr{G}$, *then so is its image interpretation* $\sigma^{\mathbf{A}'}$ *over* $\zeta(\mathscr{G})$.

Hence, we can conclude that the image of an eBH is a sub-eBA. That is,

Lemma 2.5.3 (sub-eBA and image of an eBH). *Let* $\mathbf{A}$ *and* $\mathbf{A}'$ *be eBAs,* ζ *be an eBH from* $\mathbf{A}$ *to* $\mathbf{A}'$. *Then, the image of the eBA* $\mathbf{A}$ *under the eBH* ζ *is a sub-eBA of the eBA* $\mathbf{A}'$, *i.e.* $\zeta(\mathbf{A}) \prec \mathbf{A}'$.

Proof. Obvious from Lemmas 2.5.1 and 2.5.2. $\square$

Consequently, we have that eBHs preserve sub-eBAs. Formally,

Lemma 2.5.4 (sub-eBAs preserved by eBHs). *Let* $\mathbf{A}$ *and* $\mathbf{A}'$ *be eBAs,* ζ *be an eBH from* $\mathbf{A}$ *to* $\mathbf{A}'$, *and* $\mathbf{A}''$ *be a sub-eBA of the eBA* $\mathbf{A}$ (*i.e.* $\mathbf{A}'' \prec \mathbf{A}$). *Then, the image of a sub-eBA* $\mathbf{A}''$ *is a sub-eBA of the co-domain eBA* $\mathbf{A}'$, *i.e.* $\zeta(\mathbf{A}'') \prec \mathbf{A}'$.

Next, a composition of two eBHs is a eBH. That is,

Lemma 2.5.5 (Composition of eBHs). *Let* $\mathbf{A}$, $\mathbf{A}'$, *and* $\mathbf{A}''$ *be eBAs,* ζ' *be an eBH from* $\mathbf{A}$ *to* $\mathbf{A}'$ *and* ζ *be an eBH from* $\mathbf{A}'$ *to* $\mathbf{A}''$. *Then, the composition* $\zeta \circ \zeta'$ *is an eBH from* $\mathbf{A}$ *to* $\mathbf{A}''$.

Now, two eBHs agree on the extension **ext(X)** if they agree on **X**.

Lemma 2.5.6 (*ext* and eBHs). *Let* $\mathbf{A}$ *and* $\mathbf{A}'$ *be eBAs; where* $\mathbf{A}=\langle A,\{\mathscr{F}_m \mid m \in Nat\}\rangle$, *and* $\mathbf{X}$ *be a sub-family* $\langle X,\{X_k \mid k \in Nat\}\rangle$ *of the family* $\langle A,\{\mathscr{F}_m \mid m \in Nat\}\rangle$. *Thus, if two eBHs* $\zeta,\zeta' : \mathbf{A} \to \mathbf{A}'$ *agree on* $\mathbf{X}$, *so do they on its extension* **ext(X)**.

We can generalize Lemma 2.5.6 to arbitrary finite number times of applying **ext** to **X**. Formally,

Lemma 2.5.7 (Monotonicity with *ext* and eBHs). *Let* $\mathbf{A}$ *and* $\mathbf{A}'$ *be eBAs* (*where* $\mathbf{A}=\langle A,\{\mathscr{F}_m \mid m \in Nat\}\rangle$) $\mathbf{X}$ *be a sub-family* $\langle X,\{X_k \mid k \in Nat\}\rangle$ *of the family* $\langle A,\{\mathscr{F}_m \mid m \in$

$Nat\}\rangle$. Thus, if two eBHs $\zeta', \zeta : \mathbf{A} \to \mathbf{A}'$ agree on $\mathbf{X}$, *so do they on its extension* $\mathbf{ext}^j(\mathbf{X})$ *for all* $j \geqslant 0$.

Proof. By induction on j. □

Therefore, two eBHs agree on a generated sub-eBA [**X**] if they agree on the generators **X**. Formally,

Lemma 2.5.8 (eBHs and generated sub-eBAs). *Let* **A** *and* **A**$'$ *be eBAs (where* $\mathbf{A} = \langle A, \{\mathscr{F}_m \mid m \in Nat\}\rangle$*),* **X** *be a sub-family* $\langle X, \{X_k \mid k \in Nat\}\rangle$ *of the family* $\langle A, \{\mathscr{F}_m \mid m \in Nat\}\rangle$*. Thus, if two eBHs* $\zeta' : \mathbf{A} \to \mathbf{A}'$ *and* $\zeta : \mathbf{A} \to \mathbf{A}'$ *agree on* **X**, *then so do they on the generated sub-eBA* [**X**].

Proof. By Lemma 2.5.7 and Theorem 2.4.9. □

Naturally,

Theorem 2.5.9 (Uniqueness of eBHs on the generated eBAs). *Let* **A** *be a generated eBA and* **X** *be its generator. Thus, if two eBHs* $\zeta', \zeta : \mathbf{A} \to \mathbf{A}'$ *agree on* **X**, *then they are identical, i.e.* $\zeta' = \zeta$.

Proof. By Lemma 2.5.8. □

3. Term eBA

3.1. Constructing a term eBA

In this section, we are to introduce the term eBA **T**. Apparently, the introduction is dominated by an actual construction. The key point is to build carriers $\langle T, \{\mathscr{F}_m^{\mathbf{T}} \mid m \in Nat\}\rangle$ from binding terms BT, and let the carriers form an explicitly closed family with uniform interpretations over the family. First, let us introduce an equivalence relation $\dot{\approx}$ on binding terms BT, which is essentially a version of α-conversions in Lambda calculus [2].

Definition 3.1.1 ($\dot{\approx}$ *on* BT). $\dot{\approx}$ is the least equivalence relation family (on binding terms BT) which is closed under α-conversions, ξ-conversions, anti-ξ-conversions (ξ^{-1}) and compositions of both function variables and binding operators, i.e. $\dot{\approx}$ is the least fixed point of $\mathscr{M}_{\dot{\approx}}$ where $\mathscr{M}_{\dot{\approx}}$ is defined as follows: for every equivalence relation R,

($\dot{\approx}$-α) $\langle\langle \vec{x} : t\rangle, \langle \vec{y} : t[\vec{x} := \vec{y}]\rangle\rangle \in \mathscr{M}_{\dot{\approx}}(R)$,
  where $t \in T$ and $y_j \notin FreeVar(t)(1 \leqslant j \leqslant |\vec{x}| = |\vec{y}|)$;

($\dot{\approx}$-ξ^{-1}) if $\langle\langle \vec{z} : u\rangle, \langle \vec{z'} : u'\rangle\rangle \in R$ then $\langle u[\vec{z} := \vec{y}], u'[\vec{z'} := \vec{y}]\rangle \in \mathscr{M}_{\dot{\approx}}(R)$,
  where $\{\vec{y}\} \subseteq V$ and $\{\vec{y}\} \cap ((FreeVar(u) - \{\vec{z}\}) \cup (FreeVar(u') - \{\vec{z'}\})) = \emptyset$;

($\dot{\approx}$-ξ) if $\langle t, u\rangle \in R$, then $\langle \vec{x} : t\rangle, \langle \vec{x} : u\rangle\rangle \in \mathscr{M}_{\dot{\approx}}(R)$;

($\dot{\approx}$-cmp-1) if $\langle t_j, u_j\rangle \in R$ $(1 \leqslant j \leqslant |\vec{t}| = |\vec{u}|)$, then $\langle f(\vec{t}), f(\vec{u})\rangle \in \mathscr{M}_{\dot{\approx}}(R)$;

($\dot\approx$-cmp-2) if $\langle ft_i, fu_i\rangle \in R$ $(1 \leqslant i \leqslant \ell)$ and $\langle t_j, u_j\rangle \in R$ $(1 \leqslant j \leqslant n)$, then $\langle \sigma(\vec{ft}, \vec{t}),$
$\sigma(\vec{fu}, \vec{u})\rangle \in \mathscr{M}_{\dot\approx}(R)$; where $\sigma \in \Sigma_{\langle \vec{m}, n\rangle}$ and $|\vec{m}\,| = \ell$.

Note that $\dot\approx$ is the least fixed point of $\mathscr{M}_{\dot\approx}$ on binding terms BT, i.e. $\dot\approx =_{\mathrm{df}} \bigsqcup \mathscr{M}_{\dot\approx}(\mathscr{I}) = \bigcup_{j \in Nat} \mathscr{M}^j_{\dot\approx}(\mathscr{I})$ where $\mathscr{I}$ is the family of pure *identity* relations on binding terms in BT.

In what follows, we try to establish the validity of $\dot\approx$ (Theorem 3.1.3). For this purpose, we show that syntactic substitutions are exchangeable with semantic substitutions as a first step toward the validity.

Lemma 3.1.2 (Syntactic substitution vs. semantic substitution). *For $t \in T$ and $ft \in FT_m$ $(m \in Nat)$, we have that for every environment $\langle \rho, \varphi\rangle$ on an eBA* **A**, *the following holds*:
(a) $\mathscr{A}[\![t]\!](\rho[\mathscr{A}[\![\vec{t}\,]\!](\rho, \varphi)/\vec{x}\,], \varphi[\mathscr{A}[\![\vec{fu}]\!](\rho, \varphi)/\vec{f}\,]) = \mathscr{A}[\![t\,[\vec{x}, \vec{f} := \vec{t}, \vec{fu}]\,]\!](\rho, \varphi)$,
(b) $\mathscr{A}[\![ft]\!](\rho[\mathscr{A}[\![\vec{t}]\!](\rho, \varphi)/\vec{x}], \varphi[\mathscr{A}[\![\vec{fu}]\!](\rho, \varphi)/\vec{f}\,]) = \mathscr{A}[\![ft\,[\vec{x}, \vec{f} := \vec{t}, \vec{fu}]]\!](\rho, \varphi)$,
where $\vec{f}$ and $\vec{fu}$ are a list of distinct function variables and a list of function terms with compatible arities to $\vec{f}$, respectively.

Without diverting our main interests, we refer to Section 5.1 in [29] for substitutions involved.

Proof. Combining (a) and (b) with structural induction on binding terms BT. $\square$

Theorem 3.1.3 (Validation of relation $\dot\approx$). *$\dot\approx$-equivalence is sound, i.e. $p \dot\approx q$ implies $\mathscr{A}[\![p]\!](\rho, \varphi) = \mathscr{A}[\![q]\!](\rho, \varphi)$ for every environment $\langle \rho, \varphi\rangle$.*

Proof. Induction on j of $\mathscr{M}^j_{\dot\approx}(\mathscr{I})$, since both $\mathscr{M}_{\dot\approx}$ is monotonic and its least fixed point is $\dot\approx$. $\square$

Let $[t]$ be $\{t' \in T \mid t' \dot\approx t\}$ for $t \in T$, and $[ft]$ be $\{ft' \in FT_m \mid ft' \dot\approx ft\}$ for $ft \in FT_m$. We will give the term eBA **T** below, but we avoid the terminology of "*term eBA*" before its verification.

Definition 3.1.4 (*Carriers and interpretations from BT*). Let
(i) $[T]$ be $\{[t] \mid t \in T\}$;
(ii) $g_{[ft]}([\vec{u}\,]) = [t\,[\vec{x} := \vec{u}\,]\,]$ if $ft = \langle \vec{x} : t\rangle$;
(iii) for $\sigma \in \Sigma_{\langle \vec{m}, n\rangle}$, its interpretation $\sigma^{\mathbf{T}}$ be a functional such that $\sigma^{\mathbf{T}}(\vec{g}_{[\vec{ft}]}, [\vec{t}\,]) = [\sigma(\vec{ft}, \vec{t})]$, where $\ell = |\vec{m}\,| = |\vec{ft}|$, $n = |\vec{t}\,|$ and $\vec{g}_{[\vec{ft}]}$ is the list $g_{[ft_1]}, g_{[ft_2]}, \ldots, g_{[ft_{|\vec{ft}|}]}$.

For simplicity, we sometimes refer $ft(\vec{t}\,)$ as $ft||_{|\vec{x}|}\,[\vec{x} := \vec{t}\,]$, where $\vec{x} = ft\,|_{|\vec{x}|}$ (see their definitions for $|_m$ and $||_m$ in Section 5.1 of [29], i.e. Definition 5.1.2).

Since $\dot\approx$ is an equivalence relation, this guarantees the well-definedness of Definition 3.1.4, i.e. the values do not depend on their representatives. Thus, we can also simplify $[t]$ and $g_{[ft]}$ as $\bullet_{[t]}$ and $\bullet_{[ft]}$, respectively. This is to emphasize the fact that they can be generated from the indices t (ordinary term) and ft (function term), respectively.

In what follows, we will verify that such a construction yields an eBA (Theorem 3.1.9). First, we show that the construction satisfies the extensionality (Lemma 3.1.5). It is followed by a verification of that functions are closed under compositions in the family constructed in Definition 3.1.4 (i.e. Lemma 3.1.6). Third, that the family is explicitly closed is examined (Theorem 3.1.7). Last, the uniformity of the interpretations of binding operators (BOs) is established (Lemma 3.1.8). To be explicit, let $\mathbf{T}$ be $\langle [T], \mathscr{F} \rangle$, where $\mathscr{F}$ is the family $\{\mathscr{F}_k \mid k \in Nat\}$, and $\mathscr{F}_k =_{\mathrm{df}} \{g_{[ft]} \mid ft \in FT_k\}$ for each $k \in Nat$. Thus, we have the following.

Lemma 3.1.5 ($\dot{\approx}$-*ext*). *For $ft, fu \in FT_m$ $(m \in Nat)$, if for every $[t_j]$ $(1 \leqslant j \leqslant m)$ $g_{[ft]}([\vec{t}\,]) = h_{[fu]}([\vec{t}\,])$ then $ft \dot{\approx} fu$.*

Proof. To verify this, just let t_j be $z_j \in V$ such that $z_j \notin FreeVar(ft) \cup FreeVar(fu) \cup \{\vec{z}\lceil_{j-1}\}$; and by ($\dot{\approx}$-$\xi$), ($\dot{\approx}$-$\alpha$) and transitivity, you can get it (i.e. $\dot{\approx}$-ext); where $\vec{z}\lceil_{j-1}$ is the prefix list of the list $\vec{z}$ with total $j-1$ elements. □

Next, the family $\mathbf{T}$ is closed under function compositions. Formally,

Lemma 3.1.6 (Closedness of compositions in $\mathscr{F}^{\mathbf{T}}$). *Let $g_{[ft]}$ be the function generated by $ft \in FT_m$ $(m \in Nat)$, and $\vec{g}$ be a list of functions generated by a list $\vec{ft}$ of function terms where $ft_i \in FT_k$ $(k \in Nat)$. Then, $\bullet_{[ft]} \circ \langle \vec{g} \rangle = \bullet_{[\langle \vec{x} : fu \rangle]}$, where $fu = ft||_m [\vec{y} := \vec{t}\,]$, $t_i = ft_i||_k [\vec{y}^i := \vec{x}]$, $ft_i|_k = \vec{y}^i$, and $x_j \notin (FreeVar(ft) \cup \bigcup_{1 \leqslant i \leqslant k} FreeVar(ft_i)) \cap V$ $(1 \leqslant j \leqslant |\vec{x}\,|)$.*

Proof. By $\dot{\approx}$-ext (i.e. Lemma 3.1.5). □

Theorem 3.1.7 (Explicit closedness of $\mathscr{F}^{\mathbf{T}}$). *$\mathscr{F}([T])$ is explicitly closed.*

Proof. By case analysis and Lemmas 3.1.10 and 3.1.6. □

The next lemma is about the uniformity over the family $\mathscr{F}^{\mathbf{T}}$.

Lemma 3.1.8 (Uniformity over $\mathscr{F}^{\mathbf{T}}$). *For a $\sigma \in \Sigma_{\langle \vec{m}, n \rangle}$ with $|\vec{m}\,| = \ell$, let $\vec{ft}$ be a list of function terms such that the ith element ft_i of $\vec{ft}$ has arity $k + m_i$, and all elements in list $\vec{fu}$ share a same arity k. Then,*

$$\sigma^{\mathbf{T}} \circ \langle \vec{g}, \vec{g}' \rangle = h,$$

where $\vec{x}$ are not free in $\vec{ft}$ and $|\vec{x}\,| = k$; $g_i = curry_{k,m_i}(\bullet_{[ft_i]})$ $(1 \leqslant i \leqslant \ell)$; $g'_j = \bullet_{[fu_j]}$ $(1 \leqslant j \leqslant n)$; $h = \bullet_{[\langle \vec{x} : \sigma(\vec{fv}, \vec{v}) \rangle]}$ such that $fv_i = \langle \vec{y}^i : ft_i||_{k+m_i} [\vec{z}^i := \vec{x}, \vec{y}^i\,] \rangle$, $ft_i|_{k+m_i} = \vec{z}^i$, and $y^i_j \notin FreeVar(ft_i) \cup \{\vec{x}\} \cup \{\vec{y}^i\lceil_{j-1}\}$; and $v_j = fu_j||_k [\vec{w}^j := \vec{x}\,]$ such that $fu_j|_k = \vec{w}^j$.

Proof. By Lemma 3.1.6, ($\dot{\approx}$-α), ($\dot{\approx}$-ext) and Lemma 3.1.5, $curry_{k,m}(g_{[ft]})([t_1],[t_2],\ldots,[t_k])$ can be generated by $[\langle\vec{y} : ft||_{k+m}[\vec{z}:=\vec{t},\vec{y}\,]\rangle]$, where $\vec{z}=ft|_{k+m}$ and y_j is a $y\in V$ such that $y\notin FreeVar(ft)\cup[\bigcup_{j=1}^{k} FreeVar(t_j)]\cup\{\vec{y}\lceil_{j-1}\}$. $\square$

In other words, Lemma 3.1.8 says that $\sigma^{\mathbf{T}}$ is uniform over $\mathscr{F}([T])$ for each $\sigma\in\Sigma_{\langle\vec{m},n\rangle}$. Therefore, it is appropriate to say that **T** is an eBA, and the *term eBA* will be used to refer it.

Theorem 3.1.9 (Term eBA). $\mathbf{T}=\langle[T],\mathscr{F}([T])\rangle$ *is an eBA.*

Proof. By Theorem 3.1.7 and Lemma 3.1.8. $\square$

Lemma 3.1.10 (Role of non-free variables).
(i) 1. *if* $x\notin FreeVar(t)$, $t\vec{\varrho}=t\vec{\varrho}[x:=t']$, *where* $x\in V$ *and* $t'\in T$.
   2. *if* $x\notin FreeVar(ft)$, $ft\vec{\varrho}=ft\vec{\varrho}[x:=t']$, *where* $x\in V$ *and* $t'\in T$.
(ii) 1. *for all* $m\geqslant 0$, *if* $f\notin FreeVar(t)$, $t\vec{\varrho}=t\vec{\varrho}[f:=ft']$, *where* $f\in F_m$ *and* $ft'\in FT_m$.
   2. *for all* $m\geqslant 0$, *if* $f\notin FreeVar(ft)$, $ft\vec{\varrho}=ft\vec{\varrho}[f:=ft']$, *where* $f\in F_m$ *and* $ft'\in FT_m$.

Proof. By structural induction. $\square$

3.2. Generatability of the term eBA

In this section, we show that the term eBA **T** can be generated together by its ordinary variables and function variables (Theorem 3.2.8). First we seek a syntactic counterpart $\vec{\Im}$ of **ext**. The one provided below is not a direct counterpart but it is close enough to serve our purpose.

Definition 3.2.1 (*Syntactic extension* $\vec{\Im}$). Let $\vec{X}$ be a pair $\langle X,\{X_k\mid k\in Nat\}\rangle$ where X is a subset of ordinary terms T and X_k is a subset of function terms FT_k with arity $k\geqslant 0$. We define $\vec{\Im}$ inductively as follows:

1. $\Im(\vec{X})=_{\mathrm{df}} X\cup\{ft||_m[\vec{x}:=\vec{t}\,]\mid ft\in(X_m-F_m)\wedge t_j\in X(1\leqslant j\leqslant m)\}$
$\cup\{f(\vec{t}\,)\mid f\in(X_m\cap F_m)\wedge t_j\in X(1\leqslant j\leqslant m)\}$
$\cup\{\sigma(\vec{ft},\vec{t}\,)\mid\sigma\in\Sigma_{\langle\vec{m},n\rangle}\wedge ft_i\in X_{m_i}(1\leqslant i\leqslant\ell)\wedge t_j\in X(1\leqslant j\leqslant n)\}$,
where $\ell=|\vec{m}\,|$;
2. $\Im_k(\vec{X})=_{\mathrm{df}} X_k\cup\{\langle\vec{x}:t\rangle\mid x_j\in V(1\leqslant j\leqslant k)\wedge t\in\Im(X)\}$, for $k\geqslant 0$ and $x_i\neq x_j$ $(i\neq j)$.

For convenience, let $\Im^0(\vec{X})=\vec{X}$ and $\Im^{j+1}(\vec{X})=\Im(\Im^j(\vec{X}))$ for $j\geqslant 0$. We show that $\vec{\Im}(\vec{X})$ is a proper extension of $\vec{X}$. Formally,

Lemma 3.2.2 (Boundness of $\vec{\Im}$). *For* $X\subseteq T$ *and* $X_k\subseteq FT_k$ $(k\geqslant 0)$, *we have the following*:

(a) $\vec{\Im}(\vec{X}) \subseteq T$ *and* $\vec{\Im}_k(\vec{X}) \subseteq FT_k$ *for* $k \geqslant 0$, *where the subscript* k *in* $\vec{\Im}_k(\vec{X})$ *indicates the arity*;
(b) $\vec{\Im}^j(\vec{X}) \subseteq T$ *and* $\vec{\Im}_k^j(\vec{X}) \subseteq FT_k$ $(k \geqslant 0)$, *for every* $j \geqslant 0$.

Next, we show that $\vec{\Im}$ is monotonic under the usual set inclusion order. Formally,

Lemma 3.2.3 (Monotonicity of $\vec{\Im}$). (i) $\vec{\Im}^j(\vec{X}) \subseteq \vec{\Im}^{j+1}(\vec{X})$, *for any* $j \geqslant 0$;
(ii) $\vec{\Im}^j(\vec{X}) \subseteq \vec{\Im}^{j'}(\vec{X})$, *for* $0 \leqslant j \leqslant j'$.

From Lemmas 3.2.2 and 3.2.3, we understand that the sequence $\{\vec{\Im}^j(\vec{X}) \mid j \geqslant 0\}$ has an upper bound and it is non-decreasing. Apparently, it has the least upper bound. We give it as follows.

Lemma 3.2.4 (Binding terms BT and their syntactic extensions $\vec{\Im}$). *In* $\vec{X}$, *let* $X = V$ *and* $X_k = \{\langle \vec{x} : f(\vec{x})\rangle \mid f \in F_k$ *for* $k \geqslant 0$ *and* $x_j \in V\}$, *then* $\bigcup_{j\in\omega} \vec{\Im}^j(\vec{X}) = T$ *and* $\bigcup_{j\in\omega} \vec{\Im}_k^j(\vec{X}) = FT_k$ *for* $k \geqslant 0$.

Lemmas 3.2.5 and 3.2.6 exhibit a relation from the syntactic extension $\vec{\Im}$ to the semantic extension *ext*.

Lemma 3.2.5 (From syntactic $\vec{\Im}$ to semantic *ext*). *Let* $\vec{X}$ *be the same as in Lemma* 3.2.4. *Then*,
(a) $[t] \in ext^2(\mathbf{X})$ *for* $t \in \Im(\vec{X})$;
(b) $g_{[ft]} \in ext_k^2(\mathbf{X})$ *for* $ft \in \Im_k(\vec{X})$;
where $\mathbf{X}$ *is a pair* $\langle \{\bullet_{[t]} \mid t \in X\}, \{\mathbf{X}_k \mid k \in Nat\}\rangle$ *such that* $\mathbf{X}_k = \{\bullet_{[ft]} \mid ft \in X_k\}$ *for* $k \geqslant 0$.

Proof. The proof for the first part (a) is left out. For the second part (b), i.e. $ft = \langle \vec{x} : t\rangle$ for some $t \in \Im(\vec{X})$ and $\vec{x} \in V$, there are two possibilities as follows:
(i) $t = y$ for some $y \in X$;
(ii) $t = f(\vec{y})$ for some $f \in X_m$ and $y_j \in X$ $(1 \leqslant j \leqslant m)$.

For (i), if $y = x_i \in \{\vec{x}\}$ for some $1 \leqslant i \leqslant k = |\vec{x}|$, then $g_{[\langle \vec{x}:y\rangle]} = \pi_{k,i} \in ext_k(\mathbf{X})$; if $y \notin \{\vec{x}\}$ then $g_{[\langle \vec{x}:y\rangle]} = C_{k,[y]} \in ext_k(\mathbf{X})$.

For (ii), we have $g_{[\langle \vec{x}:f(\vec{y})\rangle]} = \bullet_{[\langle \vec{z}:f(\vec{z})\rangle]} \circ \langle \bullet_{[\langle \vec{x}:y_1\rangle]}, \bullet_{[\langle \vec{x}:y_2\rangle]}, \ldots, \bullet_{[\langle \vec{x}:y_m\rangle]}\rangle$ From (i), we have that $\bullet_{[\langle \vec{x}:y_j\rangle]} \in ext_k(\mathbf{X})$, for $1 \leqslant j \leqslant m = |\vec{y}|$ and $\bullet_{[\langle \vec{z}:f(\vec{z})\rangle]} \in \mathbf{X}_m$. So, we have $g_{[\langle \vec{x}:f(\vec{y})\rangle]} \in ext_k^2(\mathbf{X})$. $\square$

The above result can be extended to the following.

Lemma 3.2.6 (Syntactic $\vec{\Im}$ and semantic *ext* with the term eBA). *Let* $\vec{X}$ *be the same as in Lemma* 3.2.4. *Then, for every* $j \geqslant 0$, *we have the following*:
(i) *if* $t \in \vec{\Im}^j(\vec{X})$, $[t] \in ext^{3j}(\mathbf{X})$;
(ii) *if* $ft \in \vec{\Im}_k^j(\vec{X})$, $g_{[ft]} \in ext_k^{3j}(\mathbf{X})$ *for* $k > 0$.

Proof. Similar to the proof of Lemma 3.2.5, we leave the first part (i) out. For the second part (ii), i.e. $ft \in \vec{\Im}_k^{j+1}(\vec{X})$, there are two possibilities: (a) $ft \in \vec{\Im}_k^{j}(\vec{X})$ and (b) $ft = \langle x_1, x_2, \ldots, x_k : t\rangle$ for some $t \in \vec{\Im}^{j+1}(\vec{X})$.

For (a), by inductive hypothesis, we have $g_{[ft]} \in ext^{3j}(\mathbf{X})$. For (b), we remind the reader that there are four cases below, which can be treated accordingly (the details of the treatment are left out): (b.1) $t \in \vec{\Im}^j(\vec{X})$, (b.2) $t = f(\vec{t}\,)$ for some $f \in F_m$ and $t_j \in \vec{\Im}^j(\vec{X})$ $(1 \leqslant j \leqslant m)$, (b.3) $t = fu||_m[\vec{y} := \vec{u}]$ for $fu \in \vec{\Im}_m^j(\vec{X})$ and $fu|_m = \vec{y}$ and $u_j \in \vec{\Im}^j(\vec{X})$, and (b.4) $t = \sigma(\vec{ft}, \vec{t}\,)$ for some $\sigma \in \Sigma_{\langle \vec{m}, n\rangle}$, $ft_i \in \vec{\Im}_{m_i}^j(\vec{X})$ and $t_j \in \vec{\Im}^j(\vec{X})$ (hint: using Lemma 3.1.8, ($\dot{\approx}$-α) and the extensionality). □

Therefore,

Lemma 3.2.7 (Term eBA and the least closure of *ext*). (i) $\bullet_{[t]} \in \bigcup_{j\in\omega} ext^j(\mathbf{X})$, *for each* $t \in T$;

(ii) $g_{[ft]} \in \bigcup_{j\in\omega} ext_k^j(\mathbf{X})$ $(k > 0)$ *for every* $ft \in FT_k$.

As a summary, we have the following.

Theorem 3.2.8 (Generated term eBA). *The term eBA is generatable from its ordinary variables and function variables, i.e.* $\mathbf{T} = [\mathbf{X}]$, *where* $\mathbf{X} = \langle V, \vec{F}\rangle$.

Proof. By Lemma 3.2.7 and the fact $[\mathbf{X}] \subseteq \mathbf{T}$. □

To conclude this section, we give an explicit connection from the semantic extension **ext** to the syntactic extension $\vec{\Im}$ below.

Lemma 3.2.9 (From the semantic *ext* to the syntactic $\vec{\Im}$). *Let* $\vec{X} = \langle X, \{X_k \mid k \in Nat\}\rangle$ *and* $\mathbf{X} = \langle \{\bullet_{[t]} \mid t \in X\}, \{\mathbf{X}_k \mid k \in Nat\}\rangle$ *where* $\mathbf{X}_k = \{\bullet_{[ft]} \mid ft \in X_k\}$ *for* $k \geqslant 0$.
(a) *For any* $\bullet_{[t]} \in ext(\mathbf{X})$, *we have that there is a* $t' \in \vec{\Im}^2(\vec{X})$ *such that* $\bullet_{[t']} = \bullet_{[t]}$.
(b) *For any* $g \in ext_k(\mathbf{X})$, *we have that there is a* $ft \in \vec{\Im}_k^2(\vec{X})$ *such that* $g = h_{[ft]}$.

This lemma can be generalized to the following.

Lemma 3.2.10 (Monotonicity from semantic *ext* to syntactic $\vec{\Im}$). *For any* $j > 0$, *we have*

(i) *if* $[t] \in ext^j(\vec{\mathbf{X}})$, *then there is a* $t' \in \vec{\Im}^{2j}(\vec{X})$ *such that* $[t] = [t']$;
(ii) *if* $g \in ext_k^j(\vec{\mathbf{X}})$, *then there is a* $ft \in \vec{\Im}_k^{2j}(\vec{X})$ *such that* $g = h_{[ft]}$.

Unlike first-order algebras, the verification that the term eBA **T** is generatable is not trivial, From this non-trivial generatability, we are ready to consider the extendability of the universal property (i.e. the freeness) from first-order algebras to *second-order* eBAs. This is the subject of next section.

4. Universal property – free eBAs

This chapter is to show whether the term eBA **T** is a *free* eBA (Theorem 4.5). If this is the case, then when $F_m = \emptyset$ for all $m \geqslant 0$, then the corresponding term eBA is an initial object in the category of eBAs (provided that there are always countably infinite ordinary variables). First, we give a definition for the "freeness" below.

Definition 4.1 (*Free eBA*). Let $\mathscr{K}$ be a class of eBAs, and let $\mathbf{A}$ be an eBA which is generated by $\mathbf{X}$, i.e. $\mathbf{A} = [\mathbf{X}]$. If for every $\mathbf{A}' \in \mathscr{K}$ and for every mapping $\zeta' : \mathbf{X} \to \mathbf{A}'$ of both a function from X to A' and functionals from X_m to $\mathscr{F}_m^{\mathbf{A}'}$ $(m \in Nat)$, there is an eBH $\zeta : \mathbf{A} \to \mathbf{A}'$ which extends ζ' (i.e. $\zeta(a) = \zeta'(a)$ for $a \in X$ and $\zeta(g) = \zeta'(g)$ for $g \in X_k$ where $k \in Nat$ and $\mathbf{X}$ is a pair of $\langle X, \{X_k \mid k \in Nat\}\rangle$), then we say $\mathbf{A}$ has the universal mapping property for $\mathscr{K}$ over $\mathbf{X}$. $\mathbf{X}$ is called a set of free generators of $\mathbf{A}$, and $\mathbf{A}$ is said to be freely generated by $\mathbf{X}$.

Theorem 4.2 (Free eBAs and the unique extension of their eBHs). *Suppose* $\mathbf{A}$ *has the universal mapping property for* $\mathscr{K}$ *over* X. *Thus, if we are given* $\mathbf{A}' \in \mathscr{K}$ *and a map* $\zeta' : \mathbf{X} \to \mathbf{A}'$, *then the extension* ζ *of the map* ζ' *such that* ζ *is an eBH from* $\mathbf{A}$ *to* $\mathbf{A}'$ *and it is unique.*

Proof. By Theorem 2.5.9. □

Next, we provide a relation between eBHs and the generated sub-eBAs.

Lemma 4.3 (eBHs and the generated sub-eBA). *Let* $\zeta' : \mathbf{A} \to \mathbf{A}'$ *be an eBH and* $\mathbf{X} \subseteq \mathbf{A}$, *then* $\zeta'([\mathbf{X}]) = [\zeta'(\mathbf{X})]$.

Proof. Just check the following: (a) $\zeta'(\mathbf{ext}(\mathbf{X})) = \mathbf{ext}(\zeta'\mathbf{X})$; (b) for all $j \geqslant 0$, $\zeta'(\mathbf{ext}^j(\mathbf{X})) = \mathbf{ext}^j(\zeta'\mathbf{X})$; (c) $\zeta'(\bigcup_{j \in Nat} \mathbf{ext}^j(\mathbf{X})) = \bigcup_{j \in Nat} \mathbf{ext}^j(\zeta'\mathbf{X})$. □

The following is to provide the uniqueness between free eBAs.

Theorem 4.4 (Isomorphism between free eBAs). *Let* $\mathbf{A}$ *and* $\mathbf{A}'$ *have the universal mapping property for* $\mathscr{K}$ *over* $\mathbf{X}$ *and* $\mathbf{X}'$, *respectively, and* $|\mathbf{X}| = |\mathbf{X}'|$. *Then, we have* $\mathbf{A} \cong \mathbf{A}'$, *where* $\mathbf{A}, \mathbf{A}' \in \mathscr{K}$, *and* $\cong$ *means "isomorphic". (Note.* $|\mathbf{X}| = |\mathbf{X}'|$ *is short for both* $|X| = |X'|$ *and* $|X_k| = |X_k'|$ *for* $k \in Nat$.)

Informally, Theorem 4.4 says that the "freeness" is unique in the sense that if two eBAs have the universal mapping property over a same class $\mathscr{K}$ of eBAs and if their bases share a same cardinality, then these two eBAs are isomorphic.

Theorem 4.5 (Freeness of the term eBA). *Let* V *be a set of ordinary variables with* $|V| = \aleph_0$, *and* $\vec{F}$ *be a family of* F_k *function variables such that* $|F_k| = \aleph_0$ *with arity*

$k \in Nat$. Then, the term eBA **T** *has the universal property for the class of all eBAs over $\langle V, \vec{F} \rangle$.*

Proof. By Theorems 4.2 and 3.2.9. □

So, the term eBA **T** is a free eBA.

5. Binding congruences (eBCs) and their quotient eBAs

For first-order algebras, a congruence is an equivalence relation preserving functionality (or compositionality). However, an extensional binding congruence (eBC) cannot be as simple as that. There are some extra requirements to be considered, e.g. constants, projections, and compositions of functions, i.e. certain care is required to deal with them. Also, since function spaces are parts of carriers, the extensionality has to be considered as well. Formally,

Definition 5.1 (*eBCs*). Let ϑ be a family of the equivalence relations on an eBA **A**. ϑ is said to be an extensional binding congruence (eBC) on **A** if ϑ satisfies the following:
(eBC-ext) $\langle g(\vec{a}), h(\vec{a}) \rangle \in \vartheta$ for every $\vec{a} \in A^m$ iff $\langle g, h \rangle \in \vartheta$, where $g, h \in \mathscr{F}_m (m \in Nat)$;
(eBC-comp-1) $\langle a_j, b_j \rangle \in \vartheta$ for each j implies $\langle g(\vec{a}), g(\vec{b}) \rangle \in \vartheta$, where $g \in \mathscr{F}_m$ and $|\vec{a}| = |\vec{b}| = m$;
(eBC-comp-2) $\langle g_i, h_i \rangle \in \vartheta$ (for every i) and $\langle a_j, b_j \rangle \in \vartheta$ (for each j) implies $\langle \sigma^{\mathbf{A}}(\vec{g}, \vec{a}), \sigma^{\mathbf{A}}(\vec{h}, \vec{b}) \rangle \in \vartheta$, where $\sigma \in \Sigma_{\langle \vec{m}, n \rangle}$ with $|\vec{m}| = \ell$, $g_i, h_i \in \mathscr{F}_{m_i} (1 \leqslant i \leqslant \ell)$ and $a_j, b_j \in A (1 \leqslant j \leqslant n)$.

Considering the analogy between eBAs and first-order algebras, a similar analogy between eBCs and first-order congruences is as follows. An eBC is a generalization of a first-order congruence, although it takes function spaces into account. However, an eBC is not as general as a first-order congruence since it cannot be as arbitrary as a first-order congruence, and it commits itself on certain primitive functions. Therefore, the well-definedness of eBCs is not obvious, and we examine it below. In other words, we are to show that an eBC preserves the function compositions (Lemma 5.2) and the uniformity (Lemma 5.5).

Lemma 5.2 (eBCs, compositions and uniformity). (a) *$\langle g, h \rangle \in \vartheta$ and $\langle g_i, h_i \rangle \in \vartheta$ (for each i) implies $\langle g(\vec{g}), h(\vec{h}) \rangle \in \vartheta$, where $g, h \in \mathscr{F}_m$ and $g_i, h_i \in \mathscr{F}_k$ for $m \in Nat$, $1 \leqslant i \leqslant m$ and $k \in Nat$;*
(b) *$\langle g_i^1, g_i^2 \rangle \in \vartheta$ (for each i) and $\langle h_j^1, h_j^2 \rangle \in \vartheta$ (for every j) implies $\langle (\sigma^{\mathbf{A}} \circ \langle \vec{g'^1}, \vec{h^1} \rangle), (\sigma^{\mathbf{A}} \circ \langle \vec{g'^2}, \vec{h^2} \rangle) \rangle \in \vartheta$ for $\sigma \in \Sigma_{\langle \vec{m}, n \rangle}$ with $|\vec{m}| = \ell$, and $k \in Nat$, where $g_i'^j = curry_{k, m_i}(g_i^j)$ and $g_i^j \in \mathscr{F}_{k+m_i}$ $(j = 1, 2)$ for $1 \leqslant i \leqslant \ell$, and $h_j^i \in \mathscr{F}_k$ $(i = 1, 2)$ for $1 \leqslant j \leqslant n$.*

Proof. For (a), it is by (eBC-ext). For (b), we know that there are three cases: (b.1) $\langle C_{m,a}, C_{m,b} \rangle \in \vartheta$ if $\langle a, b \rangle \in \vartheta$, (b.2) $\langle \pi_{m,i}, \pi_{m,i} \rangle \in \vartheta$ and (b.3) $curry_{k,m}(g)(\vec{a}) = g \circ \langle C_{m,a_1},$

$C_{m,a_2}, \ldots, C_{m,a_k}, \Pi^m_{1,m}\rangle$ for $\vec{a} \in A^k$, where $\Pi^m_{1,m}$ is the list $\pi_{m,1}$, $\pi_{m,2}, \ldots, \pi_{m,m}$. So, by (eBC-ext), we have (b). □

Similar to the first-order quotient algebras, we turn to the quotient eBAs (Definition 5.7). Firstly,

Definition 5.3 (*Quotient family*). Let ϑ be an eBC on an eBA $\mathbf{A}$. Then, $a/\vartheta =_{\mathrm{df}} \{b \in A \mid \langle a, b\rangle \in \vartheta\}$ for $a \in A$, and $g/\vartheta =_{\mathrm{df}} \{h \in \mathscr{F}_m \mid \langle g, h\rangle \in \vartheta\}$ for $g \in \mathscr{F}_m$.

It can be easily checked that $\bullet/\vartheta$ is well-defined, i.e. the values do not depend on their representatives. Furthermore, let ϑ be an eBC on an eBA $\mathbf{A}$. Then, we define the following:

(i) for $m \in Nat$, $g \in \mathscr{F}_m$ and $a_j \in A$, $\bullet_{(g/\vartheta)}(\vec{a}/\vartheta) =_{\mathrm{df}} g(\vec{a})/\vartheta$ (later, we will use g/ϑ to refer to $\bullet_{(g/\vartheta)}$ for simplicity);

(ii) for $m \in Nat$, $1 \leqslant i \leqslant m$, and $k \in Nat$, given $g \in \mathscr{F}_m$ and $h_i \in \mathscr{F}_k$, $(g/\vartheta) \circ \langle \vec{h}/\vartheta\rangle =_{\mathrm{df}} (g \circ \langle \vec{h}\rangle)/\vartheta$;

(iii) for $\sigma \in \Sigma_{\langle \vec{m}, n\rangle}$ with $|\vec{m}| = \ell$, $g_i \in \mathscr{F}_{k+m_i}$ $(1 \leqslant i \leqslant \ell)$ and $h_j \in \mathscr{F}_k$ $(1 \leqslant j \leqslant n)$, $(\sigma^{\mathbf{A}}/\vartheta) \circ \langle curry_{k,\vec{m}}(\vec{g}/\vartheta), \vec{h}/\vartheta\rangle =_{\mathrm{df}} (\sigma^{\mathbf{A}} \circ \langle curry_{k,\vec{m}}(\vec{g}), \vec{h}\rangle)/\vartheta$. It is easy to verify that the above definitions of $\bullet/\vartheta$ are well-defined, i.e. their values do not depend on their representatives.

Therefore, we can define $\mathscr{F}/\vartheta =_{\mathrm{df}} \langle A/\vartheta, \mathscr{F}_m/\vartheta_{m \in Nat}\rangle$ where $\mathscr{F}_m/\vartheta = \{g/\vartheta \mid g \in \mathscr{F}_m\}$ for $m \in Nat$. Below, we will show that $\mathscr{F}/\vartheta$ preserves the "explicitly closedness".

Lemma 5.4 (Explicit closedness of $\mathscr{F}/\vartheta$). *$\mathscr{F}/\vartheta$ is explicitly closed.*

Proof. By case analysis. □

Next, we concern of the uniformity of interpretations of BOs in $\mathscr{F}/\vartheta$.

Lemma 5.5 (Uniformity of $\mathscr{F}/\vartheta$). *For each $\sigma \in \Sigma_{\langle \vec{m}, n\rangle}$, $\sigma^{\mathbf{A}}/\vartheta$ is uniform over $\mathscr{F}/\vartheta$.*

Proof. Obvious by the definition of $\bullet/\vartheta$. □

Since $\bullet/\vartheta$ preserves both the explicit-closedness and the uniformity, we arrive at that $\mathbf{A}/\vartheta$ is an eBA. That is,

Theorem 5.6 ($\mathscr{F}/\vartheta$ and eBA). *Let $\mathbf{A}$ be an eBA and ϑ be an eBC on it. Then, $\langle A/\vartheta, \mathscr{F}/\vartheta\rangle$ is an eBA.*

Finally, we define a quotient eBA from both an eBA and its eBC. Comparing with the first-order quotient algebras, the existence of second order quotient eBAs are not as apparent as we expect.

Definition 5.7 (*Quotient eBA*). $\mathbf{A}/\vartheta$ is said to be the *quotient* eBA of an eBA $\mathbf{A}$ by an eBC ϑ iff $\mathbf{A}/\vartheta = \langle A/\vartheta, \mathscr{F}/\vartheta\rangle$ and $\mathscr{A}_\sigma^{(\mathbf{A}/\vartheta)} = \mathscr{A}_\sigma/\vartheta$.

Recall the comment after Definition 5.3; the well-definedness of $\bullet/\vartheta$ can be viewed as that for each environment $\langle\rho,\varphi\rangle$ if $\langle\mathscr{A}[\![p]\!]_{\langle\rho,\varphi\rangle},\mathscr{A}[\![q]\!]_{\langle\rho,\varphi\rangle}\rangle\in\vartheta$ then $(\mathscr{A}/\vartheta)[\![p]\!]_{\langle\rho/\vartheta,\varphi/\vartheta\rangle}=(\mathscr{A}/\vartheta)[\![q]\!]_{\langle\rho/\vartheta,\varphi/\vartheta\rangle}$. In other words, this means that the quotient eBAs preserve the satisfactions.

Let $\mathbf{A}$ be an eBA and ϑ be an eBC on $\mathbf{A}$. The natural map $v_\vartheta:\mathbf{A}\to\mathbf{A}/\vartheta$ is defined by (a) $v_\vartheta(a)=_{\mathrm{df}}a/\vartheta$, (b) $v_\vartheta(g)=_{\mathrm{df}}g/\vartheta$ and (c) $v_\vartheta(\sigma^{\mathbf{A}})=_{\mathrm{df}}\sigma^{\mathbf{A}/\vartheta}$. We are to demonstrate that the natural map is an eBH. Formally

Lemma 5.8 (Preservations of v_ϑ).
(v-cons) $v_\vartheta(C_{k,a})=C_{k,v_\vartheta(a)}$;
(v-proj) $v_\vartheta(\pi^{\mathbf{A}}_{k,i})=\pi^{(\mathbf{A}/\vartheta)}_{k,i}$ *for* $1\leqslant i\leqslant k$;
(v-cmp) $v_\vartheta(g\circ\langle\vec{h}\rangle)=v_\vartheta(g)\circ\langle v_\vartheta(\vec{h})\rangle$;
(v-unif) $v_\vartheta(\sigma^{\mathbf{A}}\circ\langle\vec{g}',\vec{h}\rangle)=\sigma^{(\mathbf{A}/\vartheta)}\circ\langle\vec{g}'',v_\vartheta(\vec{h})\rangle$ *where* $\sigma\in\Sigma_{\langle\vec{m},n\rangle}$ *with* $|\vec{m}|=\ell$, $g_i\in\mathscr{F}_{k+m_i}$, $g_i'=curry_{k,m_i}(g_i)$ *and* $g_i''=curry_{k,m_i}(v_\vartheta(g_i))$ $(1\leqslant i\leqslant\ell)$, *and* $h_j\in\mathscr{F}_k$ $(1\leqslant j\leqslant n)$.

Proof. By the extensionality. □

Therefore,

Theorem 5.9 (v_ϑ and eBH). *Let ϑ be an eBC on an eBA* $\mathbf{A}$. *Then, the natural map v_ϑ from* $\mathbf{A}$ *to* $\mathbf{A}/\vartheta$ *is an onto (surjective) eBH.*

Proof. By Lemma 5.8. □

Later, the natural map v_ϑ is referred to as the *natural* eBH associated with the eBC ϑ. Analogous to first-order algebras [5], we define a new eBC over a quotient eBA as follows. Suppose ϑ and ϑ' are eBCs on $\mathbf{A}$ and $\vec{\vartheta}\subseteq\vec{\vartheta}'$, i.e. $\vartheta\subseteq\vartheta'$ and $\vartheta_m\subseteq\vartheta'_m$ for $m\in Nat$. Later, we will simply refer to them as $\vartheta\subseteq\vartheta'$. Then, we define a double eBC as follows. Let $\vartheta'/\vartheta=_{\mathrm{df}}\{\langle g/\vartheta,h/\vartheta\rangle\mid\langle g,h\rangle\in\vartheta'\}$. We have Lemma 5.10 and Theorem 5.11.

Lemma 5.10 (Double eBC). *If ϑ and ϑ' are eBCs on* $\mathbf{A}$ *and $\vartheta\subseteq\vartheta'$, then ϑ'/ϑ is an eBC on* $\mathbf{A}/\vartheta$.

Informally, this lemma says that a double eBC ϑ'/ϑ is an eBC on quotient eBA $\mathbf{A}/\vartheta$. Furthermore, from a double eBC we can obtain a double quotient eBA. The relationship among these eBAs, their quotient eBAs and their double quotient eBAs is as follows.

Theorem 5.11 (Double quotient eBAs). *If ϑ' and ϑ are eBCs on* $\mathbf{A}$ *and $\vartheta\subseteq\vartheta'$, then the map $\zeta':(\mathbf{A}/\vartheta)/(\vartheta'/\vartheta)\to\mathbf{A}/\vartheta'$, defined by $\zeta'((a/\vartheta)/(\vartheta'/\vartheta))=_{\mathrm{df}}a/\vartheta'$ and $\zeta'((g/\vartheta)/(\vartheta'/\vartheta))=_{\mathrm{df}}g/\vartheta'$, is an extensional binding isomorphism (i.e. a bijective eBH or an isomorphic eBH).*

Considering the quotient eBAs, we relate the generated eBAs with their quotient eBAs below. That is,

Theorem 5.12 (Generated eBAs and their quotient eBAs). *Let* $\mathbf{A}$ *be an eBA generated by* $\mathbf{X}$. *Thus, if* ϑ *is an eBC on* $\mathbf{A}$, *then its quotient eBA* $\mathbf{A}/\vartheta$ *can be generated by* $\mathbf{X}/\vartheta$ (*i.e.* $\mathbf{A}/\vartheta=[\mathbf{X}/\vartheta]$).

Proof. By Theorem 5.9, we know that the natural v_ϑ is an eBH. Hence, by Lemma 4.3, we have that $\mathbf{A}/\vartheta$ is generated by $\mathbf{X}/\vartheta$. □

From first-order algebras, we understand that the purpose of introducing congruences and their quotient algebras is to provide a general framework to work on kernels of homomorphisms. This is a crucial step toward the success of Birkhoff's approach. We intend to follow this in what follows.

A first-order congruence plays two roles. They are (a) an equivalence relation preserving compositionality, and (b) yielding an quotient algebra. However, these two roles are tied to each other in first-order case. The situation is different for eBAs, i.e. these two roles are separated. Corresponding to (a), there is a concept of the *binding core* of an eBH; and the *binding kernel* of an eBH corresponds to (b). Informally, a pair of elements are in the binding core of an eBH iff they share a same image under the eBH. Formally,

Definition 5.13 (*Binding core* $\vec{\nabla}_\zeta$). Let $\zeta:\mathbf{A}\rightarrow\mathbf{A}'$ be an eBH. Then, the binding core of ζ (the core of ζ for short), written as $\vec{\nabla}_\zeta$, is a pair $\langle\nabla,\{\nabla_k \mid k\in Nat\}\rangle$ where ∇ is a relation on the ordinary objects of the eBA $\mathbf{A}$ and ∇_k is a relation on the functions of the eBA $\mathbf{A}$ with arity $k\in Nat$ such that $\nabla=_{\mathrm{df}}\{\langle a,b\rangle\in A\times A \mid \zeta(a)=\zeta(b)\}$ and $\nabla_k=_{\mathrm{df}}\{\langle g,h\rangle\in\mathscr{F}_k\times\mathscr{F}_k \mid \zeta(g)=\zeta(h)\}$.

However, the above basic requirement on a relation obtained from an eBH is not enough to be the binding kernel of the eBH. The reason for this is the extensionality of eBCs. We introduce kernels of eBHs as follows.

Definition 5.14 (*Binding kernel of an eBH*). Let $\zeta:\mathbf{A}\rightarrow\mathbf{A}'$ be an eBH. The binding kernel of ζ (or simply kernel of ζ), written as $Ker(\zeta)$, is defined as the least pair $\langle R,\{R_k \mid k\in Nat\}\rangle$ on an eBA $\mathbf{A}$ such that the following hold:
(a) R is an equivalence relation on ordinary objects (i.e. $R\subseteq A\times A$);
(b) R_k is an equivalence relation on functions with arity k (i.e. $R_k\subseteq\mathscr{F}_k^{\mathbf{A}}\times\mathscr{F}_k^{\mathbf{A}}$);
(c) $\vec{\nabla}_\zeta\subseteq\vec{R}$, i.e. both $\nabla\subseteq R$ and $\nabla_k\subseteq R_k$ with $k\in Nat$; and
(d) $\mathbf{ker}(\vec{R})\subseteq\vec{R}$, where $\mathbf{ker}$ is defined as follows:

$$ker(\vec{R}')=R'\cup\{\langle g(\vec{a}),h(\vec{b})\rangle \mid \langle g,h\rangle\in R'_k \text{ and } \langle a_j,b_j\rangle\in R'\}$$

$$\cup\{\langle\sigma^{\mathbf{A}}(\vec{g},\vec{a}),\sigma^{\mathbf{A}}(\vec{h},\vec{b})\rangle|\langle g_i,h_i\rangle\in R'_{m_i} \text{ and } \langle a_j,b_j\rangle\in R'\}$$

and

$$ker_k(\vec{R'}) = R'_k \cup \{\langle g,h\rangle \in \mathscr{F}_k^2 \mid \langle g(\vec{a}), h(\vec{b})\rangle \in R' \text{ for every } \langle a_j, b_j\rangle \in R'\}$$

for any relation $\vec{R'}$ on $\mathbf{A}$.

To see the difference between $\vec{\nabla}_\zeta$ and $Ker(\zeta)$, we have the following two facts: let $\mathbf{A}$ and $\mathbf{A}'$ be eBAs, $\zeta : \mathbf{A} \to \mathbf{A}'$ be an eBH and $g, g' \in \mathscr{F}_m$.
(1) $\langle g, g'\rangle \in \vec{\nabla}_\zeta$ implies $\langle g(\vec{a}), g'(\vec{a})\rangle \in \vec{\nabla}_\zeta$ for every $\vec{a} \in A^m$, and
(2) $\langle g, g'\rangle \in Ker(\zeta)$ iff $\langle g(\vec{a}), g'(\vec{a})\rangle \in Ker(\zeta)$ for every $\vec{a} \in A^m$;

For a concrete example, let $\Sigma_{\langle \vec{m}, n\rangle} = \emptyset$ when $\vec{m} \neq \varepsilon$ (i.e. essentially a first-order signature); one can take $\mathbf{A} = \langle \mathscr{F}, \mathscr{A}\rangle$ to be an eBA with the signature such that $\mathscr{A}$ is the interpretation of BOs and $\mathscr{F} = \langle A, \{\mathscr{F}_m \mid m \in Nat\}\rangle$ where $A = \{a, b, c\}$ and $\mathscr{F}_m$ is some part of full function space from A^m to A closed under constant functions, projections and compositions with arity $m \geqslant 0$; and let $\mathbf{A}'$ be another identical eBA with $\mathbf{A}$ and an eBH ζ is a family of both a function from A to A' and functionals from $\mathscr{F}_m$ to $\mathscr{F}'_m$ for $m \in Nat$. More specifically,

(i) the "*object*" function ζ maps both a and c to a, and it maps b to b, i.e.

$$\zeta(\bullet) = \begin{cases} a & \text{if } \bullet = a, \\ b & \text{if } \bullet = b, \\ a & \text{if } \bullet = c. \end{cases}$$

and

(ii) the "*function*" functional ζ_m from $\mathscr{F}_m$ to $\mathscr{F}'_m$ is almost the *identity* functional between these two function spaces except that functions like constant function $C_{m,c}$ whose image is constant function $C_{m,a}$ not $C_{m,c}$ for $m \in Nat$.

Since the number of functions in the full function space of $A^m \to A$ is 3^{3^m} with $m \in Nat$, i.e. 27 if $m = 1$ (see Table 1), we only give the details of both ∇_ζ and Ker_ζ for the case $m = 1$. Note that to preserve the compositionality of ζ both $\mathscr{F}_1(A)$ and $\mathscr{F}_1'(A')$ have, at most, 15 functions in total; i.e. they are

$$\{id, f_1, f_2, f_6, f_7, C_{1,c}, C_{1,b}, g_4, g_5, h_0, C_{1,a}, h_2, h_6, h_7, h_8\};$$

otherwise there is a contradiction for ζ being an eBH, e.g. consider g_0 in Table 1 which does not belong to the above collection because of the following:

$$\zeta(g_0)(a) = \zeta(g_0)(\zeta(a)) = \zeta(g_0(a)) = \zeta(a) = a$$

and

$$\zeta(g_0)(a) = \zeta(g_0)(\zeta(c)) = \zeta(g_0(c)) = \zeta(b) = b.$$

On the other hand, this case of g_0 can be regarded as a justification for working on an explicitly closed family of functions in eBAs instead of their full function spaces. Also, this situation will not appear if the eBA $\mathbf{A}$ is the term eBA $\mathbf{T}$ regardless of what the other eBA $\mathbf{A}'$ is (guess why?).

Table 1
Full function space of $\{a,b,c\} \to \{a,b,c\}$

$id(x)=\begin{cases} a & \text{if } x=a \\ b & \text{if } x=b \\ c & \text{if } x=c \end{cases}$	$g_0(y)=\begin{cases} a & \text{if } y=a \\ b & \text{if } y=b \\ b & \text{if } y=c \end{cases}$	$h_0(z)=\begin{cases} a & \text{if } z=a \\ b & \text{if } z=b \\ a & \text{if } z=c \end{cases}$
$f_1(x)=\begin{cases} a & \text{if } x=a \\ a & \text{if } x=b \\ c & \text{if } x=c \end{cases}$	$g_1(y)=\begin{cases} a & \text{if } y=a \\ a & \text{if } y=b \\ b & \text{if } y=c \end{cases}$	$C_{1,a}(z)=\begin{cases} a & \text{if } z=a \\ a & \text{if } z=b \\ a & \text{if } z=c \end{cases}$
$f_2(x)=\begin{cases} a & \text{if } x=a \\ c & \text{if } x=b \\ c & \text{if } x=c \end{cases}$	$g_2(y)=\begin{cases} a & \text{if } y=a \\ c & \text{if } y=b \\ b & \text{if } y=c \end{cases}$	$h_2(z)=\begin{cases} a & \text{if } z=a \\ c & \text{if } z=b \\ a & \text{if } z=c \end{cases}$
$f_3(x)=\begin{cases} b & \text{if } x=a \\ b & \text{if } x=b \\ c & \text{if } x=c \end{cases}$	$C_{1,b}(y)=\begin{cases} b & \text{if } y=a \\ b & \text{if } y=b \\ b & \text{if } y=c \end{cases}$	$h_3(z)=\begin{cases} b & \text{if } z=a \\ b & \text{if } z=b \\ a & \text{if } z=c \end{cases}$
$f_4(x)=\begin{cases} b & \text{if } x=a \\ a & \text{if } x=b \\ c & \text{if } x=c \end{cases}$	$g_4(y)=\begin{cases} b & \text{if } y=a \\ a & \text{if } y=b \\ b & \text{if } y=c \end{cases}$	$h_4(z)=\begin{cases} b & \text{if } z=a \\ a & \text{if } z=b \\ a & \text{if } z=c \end{cases}$
$f_5(x)=\begin{cases} b & \text{if } x=a \\ c & \text{if } x=b \\ c & \text{if } x=c \end{cases}$	$g_5(y)=\begin{cases} b & \text{if } y=a \\ c & \text{if } y=b \\ b & \text{if } y=c \end{cases}$	$h_5(z)=\begin{cases} b & \text{if } z=a \\ c & \text{if } z=b \\ a & \text{if } z=c \end{cases}$
$f_6(x)=\begin{cases} c & \text{if } x=a \\ b & \text{if } x=b \\ c & \text{if } x=c \end{cases}$	$g_6(y)=\begin{cases} c & \text{if } y=a \\ b & \text{if } y=b \\ b & \text{if } y=c \end{cases}$	$h_6(z)=\begin{cases} c & \text{if } z=a \\ b & \text{if } z=b \\ a & \text{if } z=c \end{cases}$
$f_7(x)=\begin{cases} c & \text{if } x=a \\ a & \text{if } x=b \\ c & \text{if } x=c \end{cases}$	$g_7(y)=\begin{cases} c & \text{if } y=a \\ a & \text{if } y=b \\ b & \text{if } y=c \end{cases}$	$h_7(z)=\begin{cases} c & \text{if } z=a \\ a & \text{if } z=b \\ a & \text{if } z=c \end{cases}$
$C_{1,c}(x)=\begin{cases} c & \text{if } x=a \\ c & \text{if } x=b \\ c & \text{if } x=c \end{cases}$	$g_8(y)=\begin{cases} c & \text{if } y=a \\ c & \text{if } y=b \\ b & \text{if } y=c \end{cases}$	$h_8(z)=\begin{cases} c & \text{if } z=a \\ c & \text{if } z=b \\ a & \text{if } z=c \end{cases}$

Since ζ is the "identity" on $\mathscr{F}_m \times \mathscr{F}'_m$ (where $\mathscr{F}_m = \mathscr{F}'_m$), ∇_1 is the "identity" relation on $\mathscr{F}_1$ but Ker_1 properly contains ∇_1 with extra

$$\left\{ \begin{array}{lllll} \langle C_{1,a}, f_1\rangle, & \langle C_{1,a}, f_2\rangle, & \langle C_{1,a}, f_7\rangle, & \langle id, f_6\rangle, & \langle id, h_0\rangle \\ \langle C_{1,a}, h_2\rangle, & \langle id, h_6\rangle, & \langle C_{1,a}, h_7\rangle, & \langle C_{1,a}, h_8\rangle, & \langle g_4, g_5\rangle \end{array} \right\}$$

and their reflexive and transitive closure. Interested readers are encouraged to work out the rest and compare the difference between binding core $\vec{\nabla}_\zeta$ and binding kernel $Ker(\zeta)$.

Furthermore, keeping our previous analogy between eBAs and first-order algebras, we know that the binding core and the binding kernel becomes almost the same if

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\nu_\zeta"] \arrow[dr, "\zeta"'] & \mathbf{A}/Ker(\zeta) \arrow[d, "\hat{\zeta}"] \\ & \mathbf{A}' \end{tikzcd}$$

Fig. 1. Commutativity.

the signature Σ has the property $\Sigma_{\langle \vec{m},n \rangle} = \emptyset$ for every $\vec{m} \neq \varepsilon$. The possible difference is on the carriers of function spaces. If we disregard this, then the binding core and the binding kernel are the same.

Anyway, a first-order kernel is a first-order congruence and yields a first-order quotient algebra. In contrast, a binding core is not necessarily an eBC, but a binding kernel is, and it yields a quotient eBA (Theorem 5.16). The next lemma is to justify this point of view. However, let us verify that Definition 5.14 yields an eBC (Theorem 5.15).

Theorem 5.15 (Kernel and eBC). *Let $\zeta : \mathbf{A} \to \mathbf{A}'$ be an eBH. Then, $Ker(\zeta)$ is an eBC on* $\mathbf{A}$.

Theorem 5.16 (Kernels and their corresponding quotient eBAs). *Let $\zeta : \mathbf{A} \to \mathbf{A}'$ be an onto (surjective) eBH. Then, there is an isomorphic eBH $\hat{\zeta} : \mathbf{A}/Ker(\zeta) \to \mathbf{A}'$ such that $\zeta = \hat{\zeta} \circ \nu_\zeta$ where $\nu_\zeta =_{\mathrm{df}} \nu_{Ker(\zeta)}$ (the natural eBH associated with the kernel $Ker(\zeta)$). That is, the diagram in Fig.* 1 *commutes.*

Proof. The key point of the proof is the coincidence between the core ∇_ζ and the kernel $Ker(\zeta)$. This coincidence is because of that ζ is an onto eBH. The rest is just a routine check. $\square$

Now, we come to the key issue of Birkhoff's approach, i.e. whether the "*onto*" condition on ζ can be withdrawn from Theorem 5.16. To seek a general answer to this question, we are led to the *admissibility*, which is the subject of next Section.

6. Admissible freeness

Referring to Theorem 5.16, can we weaken the condition "*onto*" so that we still have the diagram (Fig. 1) commutes? To answer this question, we introduce a concept of the "*admissibility*". The essence of the admissibility is the extensionality.

Definition 6.1 (*Admissible eBHs*). Let $\zeta : \mathbf{A} \to \mathbf{A}'$ be an eBH. Thus, ζ is *admissible* iff the image of the eBA A under the eBH ζ is a perfect sub-eBA of the eBA $\mathbf{A}'$, i.e. $\zeta(\mathbf{A}) \preccurlyeq \mathbf{A}'$.

In other words, the eBH $\zeta : \mathbf{A} \to \mathbf{A}'$ is admissible, sometimes written as $\zeta : \mathbf{A} \dot{\to} \mathbf{A}'$ (the dot $\cdot$ on top of $\to$ shows the admissibility) iff the eBH ζ preserves the extensionality in its image. The following is the introduction of the "*admissible freeness*".

Definition 6.2 (*Admissibly free eBA*). Let $\mathscr{K}$ be a class of eBAs, and $\mathbf{U}$ be an eBA generated by $\mathbf{X}$. Thus,

(a) suppose that a map $\zeta : \mathbf{X} \to \mathbf{A}'$ is a family of both a function from X to A' and functionals from X_k to $\mathscr{F}_k'$ for $k \in Nat$, the map ζ is said to be *admissible* if the generated sub-eBA $[\zeta(\mathbf{X})]$ from the image of X under the eBH ζ is a perfect sub-eBA of the eBA $\mathbf{A}'$, i.e. $[\zeta(\mathbf{X})] \preccurlyeq \mathbf{A}'$, where $\mathbf{A}'$ is an eBA;

(b) for every $\mathbf{A}' \in \mathscr{K}$ and if for every admissible map $\zeta : \mathbf{X} \to \mathbf{A}'$, there is an eBH $\zeta' : \mathbf{U} \to \mathbf{A}'$ which extends the map ζ (i.e. the map ζ and the eBH ζ' agree on $\mathbf{X}$), then, we say that $\mathbf{U}$ has the *admissible universal mapping property* for $\mathscr{K}$ over $\mathbf{X}$. $\mathbf{X}$ is said to be the admissible free generators of $\mathbf{U}$ and $\mathbf{U}$ is said to be an *admissible-freely generated* by $\mathbf{X}$.

By Theorem 2.5.9, for an admissibly free eBA, we have the following: suppose $\mathbf{U}$ has the admissible universal mapping property for $\mathscr{K}$ over $\mathbf{X}$; thus, if given $\mathbf{A}' \in \mathscr{K}$ and an admissible map $\zeta : \mathbf{X} \to \mathbf{A}'$, then there exists an unique extension ζ' of the map ζ such that ζ' is an eBH $\zeta' : \mathbf{U} \to \mathbf{A}'$ and they (ζ and ζ') agree on $\mathbf{X}$.

Considering different admissibly-free eBAs, we have the following.

Theorem 6.3 (Isomorphism between admissibly free eBAs). *Let* $\mathbf{U}$ *and* $\mathbf{U}'$ *have the admissible universal mapping property for* $\mathscr{K}$ *over* $\mathbf{X}$ *and* $\mathbf{X}'$, *respectively; and let* $|\mathbf{X}| = |\mathbf{X}'|$ *(i.e.* $|X| = |X'|$ *and* $|X_k| = |X_k'|$ *for* $k \in Nat$*), then* $\mathbf{U}$ *and* $\mathbf{U}'$ *are isomorphic, i.e.* $\mathbf{U} \cong \mathbf{U}'$ *where* $\mathbf{U}, \mathbf{U}' \in \mathscr{K}$.

The admissibility makes the difference between the binding core and the binding kernel disappear. In other words, it means that the "admissibility" eliminate the difference between binding kernels and first-order kernels, see Lemma 6.4.

Lemma 6.4 (Coincidence of kernels and cores of admissible eBHs). *Let* $\zeta : \mathbf{A} \to \mathbf{A}'$ *be an admissible eBH, then* $Ker(\zeta) = \nabla_\zeta$.

(A proof hint: $\vec{\nabla}_\zeta$ is an fixpoint of **ker** in Definition 5.14 under the admissible condition)

Some examples of eBHs which have the coincidences between binding kernels and binding cores are: either (i) these eBHs with surjective images on their base level, or (ii) these eBHs with the one-to-one (i.e. 1–1 or injective) mapping on their base level.

Considering the binding kernel of an admissible eBH and its quotient eBA, we have the following.

Theorem 6.5 (Kernel of an admissible eBH and its quotient eBA). *Let $\zeta : \mathbf{A} \to \mathbf{A}'$ be an admissible eBH. Then, there is a one-to-one (i.e. 1–1 or injective) eBH $\hat{\zeta} : \mathbf{A}/Ker(\zeta) \to \mathbf{A}'$ such that $\zeta = \hat{\zeta} \circ \nu_\zeta$ (see Fig. 1).*

Proof. Define $\zeta(g/Ker(\zeta)) =_{\text{df}} \zeta(g)$. By Lemma 6.4, it is well-defined and is a 1–1 eBH. □

The importance of the admissibility is established by a few subsequent results, which are summarized as one important fact, i.e. Theorem 6.6 is the best result we can have in Birkhoff's approach. Recall that the essence of Birkhoff's method is the equivalences among the following three (1) satisfactions of algebras, (2) kernels of the algebras and (3) satisfactions of the corresponding quotient term algebras. We proceed as follows.

Lemma 6.6 (Injectiveness of eBH $\hat{\zeta}$). *Let $\zeta : \mathbf{A} \to \mathbf{A}'$ be an eBH and $\hat{\zeta} : \mathbf{A}/Ker(\zeta) \to \mathbf{A}'$ be another eBH such that $\zeta = \hat{\zeta} \circ \nu_\zeta$; then, $\hat{\zeta}$ must be 1–1 (see Fig. 1).*

Proof. Let $g'/Ker(\zeta), h'/Ker(\zeta) \in \mathbf{A}/Ker(\zeta)$ and $\hat{\zeta}(g'/Ker(\zeta)) = \hat{\zeta}(h'/Ker(\zeta))$. Since (the natural eBH associated with the kernel of the eBH ζ) ν_ζ is onto, there exists $g, h \in \mathbf{A}$ such that $\nu_\zeta(g) = g'/Ker(\zeta)$ and $\nu_\zeta(h) = h'/Ker(\zeta)$. Therefore, for every such a pair of g and h, we have $\hat{\zeta} \circ \nu_\zeta(g) = \hat{\zeta} \circ \nu_\zeta(h)$, i.e. $\zeta(g) = \zeta(h)$. So, $\langle g, h \rangle \in \nabla_\zeta \subseteq Ker(\zeta)$, i.e. $g'/Ker(\zeta) = h'/Ker(\zeta)$. □

See Fig. 1, this lemma implies that if the diagram commutes then the eBH starting from the quotient eBA must be injective. Consequently, the original eBH which we start with, must be admissible, see Corollary 6.7.

Corollary 6.7 (Admissibility and commutativity). *Let $\zeta : \mathbf{A} \to \mathbf{A}'$ be an eBH and $\hat{\zeta} : \mathbf{A}/Ker(\zeta) \to \mathbf{A}'$ be another eBH such that $\zeta = \hat{\zeta} \circ \nu_\zeta$; then, the eBH ζ is admissible (see Fig. 1).*

Proof. By Lemma 6.6, the image of the eBH $\hat{\zeta}$ is a perfect sub-eBA. So is the image of the eBH ζ. □

From Theorem 6.5 and Corollary 6.7, we understand that the admissibility is a *necessary and sufficient* condition for the diagram (Fig. 1) to commute. Because of this, we claim that the admissible completeness (Corollary 8.11) is the best result in Birkhoff's approach for eBAs. However, some potential improvements are discussed in Section 9.3.

Theorem 6.8 (Admissibility and double quotient eBA). *Let $\zeta' : \mathbf{A} \to \mathbf{A}'$ and $\zeta : \mathbf{A} \to \mathbf{A}''$ be eBHs and $Ker(\zeta) \subseteq Ker(\zeta')$ such that the eBH ζ' is admissible and the eBH ζ is*

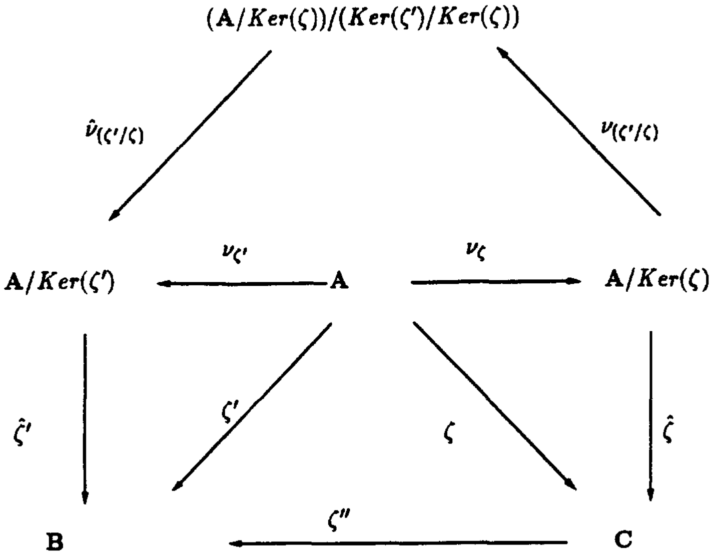

Fig. 2. Quotient eBA and double quotient eBA.

onto; then, there exists a new eBH $\zeta'' : \mathbf{A}'' \to \mathbf{A}'$ *such that these three eBHs commutes, i.e.* $\zeta' = \zeta'' \circ \zeta$.

Proof. See Fig. 2, and we define $\zeta'' =_{\mathrm{df}} \hat{\zeta}' \circ \hat{\nu}_{(\zeta'/\zeta)} \circ \nu_{(\zeta'/\zeta)} \circ \hat{\zeta}^{-1}$, where $\hat{\zeta}'$ is a 1–1 eBH by Theorem 6.5. $\hat{\zeta}^{-1}$ is an isomorphic eBH by Theorem 5.16. $\hat{\nu}_{(\zeta'/\zeta)}$ is an isomorphic eBH by Lemma 5.11. □

The following two lemmas provide the fact that eBCs are closed under intersections.

Lemma 6.9 (Intersection of two eBCs). *Let* ϑ' *and* ϑ *be eBCs on an eBA* $\mathbf{A}$. *Then, their intersection* $\vartheta' \cap \vartheta$ *is also an eBC on* $\mathbf{A}$.

Consequently,

Lemma 6.10 (Intersection of eBCs). *Let* Ψ *be a class of eBCs on* $\mathbf{A}$. *Then, its intersection* $\cap \Psi$ *is an eBC on* $\mathbf{A}$.

As a result of both Theorem 6.5 and Corollary 6.7, we define the *admissible* eBCs with respect to either an eBA or a collection of eBAs as follows.

Definition 6.11 (*Admissible eBC* $\dot{\vartheta}$). Let $\mathbf{A}$ be an eBA and $\mathscr{K}$ be a class of eBAs. Thus,

(a) the admissible eBC $\dot{\vartheta}_{\mathbf{A}}$ on the term eBA $\mathbf{T}$ with respect to the eBA $\mathbf{A}$ is defined to be the intersection $\bigcap_{\zeta:\mathbf{T}\to\mathbf{A}} Ker(\zeta)$;

$$\begin{tikzcd} \mathbf{X} \arrow[r, "\nu_{\dot{\vartheta}_{\mathcal{K}}}\lceil_{\mathbf{X}}"] \arrow[dr, "\zeta \circ \nu_{\dot{\vartheta}_{\mathcal{K}}}\lceil_{\mathbf{X}}"'] & \dot{\mathbf{X}} \arrow[d, "\zeta"] \\ & \mathbf{A} \in \mathcal{K} \end{tikzcd}$$

Fig. 3. Admissible environment and admissibly freeness.

(b) the admissible BC $\dot{\vartheta}_{\mathscr{K}}$ on the term eBA $\mathbf{T}$ with respect to the class $\mathscr{K}$ is defined to be the intersection $\bigcap_{\mathbf{A} \in \mathscr{K}} \dot{\vartheta}_{\mathbf{A}}$.

Next, we give two conditions under which the composition of two admissible eBHs is admissible (Lemma 6.12); or more precisely under which the admissibility is preserved by a composition of two admissible eBHs.

Lemma 6.12 (Admissibility under compositions). *Let $\zeta' : \mathbf{A} \to \mathbf{A}'$ be admissible (or onto) eBH and $\zeta : \mathbf{A}' \to \mathbf{A}''$ be 1–1 (or admissible) eBH; then, their composition $\zeta \circ \zeta' : \mathbf{A} \to \mathbf{A}''$ is an admissible eBH at either case*

Theorem 6.13 (Admissibly free quotient term eBA). *Let $\dot{\mathbf{T}}_{\mathscr{K}}(\dot{\mathbf{X}})$ be the quotient eBA $\mathbf{T}(\mathbf{X})/\dot{\vartheta}_{\mathscr{K}}$ where $\dot{\mathbf{X}} = \mathbf{X}/\dot{\vartheta}_{\mathscr{K}}$; then, it has the admissible universal mapping property for $\mathscr{K}$ over $\dot{\mathbf{X}}$ where $\mathscr{K}$ is a class of eBAs.*

Proof. (1) By both Theorem 5.12 and Lemma 4.3, the quotient eBA $\dot{\mathbf{T}}_{\mathscr{K}}(\dot{\mathbf{X}})$ can be generated by $\dot{\mathbf{X}}$. (2) Let ζ be the admissible eBH as in Fig. 3. We have three cases (2.a), (2.b) and (2.c) below.

(2.a) By Theorem 4.3, the composition $\zeta \circ v_{\dot{\vartheta}_{\mathscr{K}}} \lceil_{\mathbf{X}}$ can be uniquely extended to an eBH $\zeta' : \mathbf{T} \to \mathbf{A}$. (2.b) The admissibility of the eBH ζ is preserved by the uniquely-extended eBH ζ' because of that ζ' is admissible can be obtained by a similar proof of Lemma 6.12 where the eBH ζ is admissible and the natural eBH $v_{\dot{\vartheta}_{\mathscr{K}}}$ is onto. (2.c) Hence, by Theorems 6.5 and 6.8, there exists an eBH $\hat{\zeta}' : \dot{\mathbf{T}}_{\mathscr{K}}(\dot{X}) \to \mathbf{A}$ such that $\zeta' = \hat{\zeta}' \circ v_{\dot{\vartheta}_{\mathscr{K}}}$ where $\hat{\zeta}' \lceil_{\dot{\mathbf{X}}} = \zeta$, since $Ker(v_{\dot{\vartheta}_{\mathscr{K}}}) = \dot{\vartheta}_{\mathscr{K}} \subseteq Ker(\zeta')$ and the natural eBH $v_{\dot{\vartheta}_{\mathscr{K}}}$ is onto.

(3) $\hat{\zeta}'(\dot{\mathbf{X}}) = \hat{\zeta}' \circ v_{\dot{\vartheta}_{\mathscr{K}}}(\mathbf{X}) = \zeta'(\mathbf{X}) = \zeta \circ v_{\dot{\vartheta}_{\mathscr{K}}} \lceil_{\mathbf{X}}(\mathbf{X}) = \zeta(\dot{\mathbf{X}})$. □

Although the quotient term eBA $\dot{\mathbf{T}}$ is not a free eBA, it is *admissibly free*. Nevertheless, combining both Theorem 6.5 and Corollary 6.7, it implies that the quotient term eBA $\dot{\mathbf{T}}$ is admissibly free cannot be improved to (totally) free in Birkhoff's approach.

The central idea of Birkhoff's approach is to capture the equality by the kernels. This can be done in first-order algebras simply because both the cores and kernels coincide with each other. This coincidence does not hold in general for eBAs. Therefore, we turn to introduce *admissible binding equations* instead of *binding equations* (BEs). This is the subject of next section.

7. Admissible binding equations

Analogous to first-order algebras (see [5] or [29, Ch. 2]) we can define the following:

(a) $\mathbf{A} \models_{\mathrm{eBA}} p \simeq q$ (or $\mathbf{A} \models p \simeq q$ for short) iff $\mathscr{A}[\![p]\!](\vec{\rho}) = \mathscr{A}[\![q]\!](\vec{\rho})$ for all environment $\langle \rho, \varphi \rangle$;

(b) a class $\mathscr{K}$ of eBAs satisfies $p \simeq q$, written as $\mathscr{K} \models_{\mathrm{eBA}} p \simeq q$ or simply $\mathscr{K} \models p \simeq q$, iff each member $\mathbf{A}$ of $\mathscr{K}$ satisfies $p \simeq q$, i.e. $\mathbf{A} \models p \simeq q$ for every $\mathbf{A} \in \mathscr{K}$;

(c) let Γ_{BE} (or Γ as an abbreviation) be a set of binding equations (BEs), we say that $\mathscr{K}$ satisfies Γ_{BE}, written as $\mathscr{K} \models_{\mathrm{eBA}} \Gamma_{\mathrm{BE}}$ (or simply $\mathscr{K} \models \Gamma$), iff $\mathscr{K} \models p \simeq q$ for each $p \simeq q \in \Gamma_{\mathrm{BE}}$; we also use $Mod_{\Sigma}(\Gamma_{\mathrm{BE}})$ (or simply $Mod(\Gamma)$) to denote the class of the eBAs whose member $\mathbf{A}$ satisfies Γ_{BE};

(d) $\Gamma_{\mathrm{BE}} \models_{\mathrm{eBA}} p \simeq q$ (or in short $\Gamma \models p \simeq q$) iff $\mathbf{A} \models \Gamma_{\mathrm{BE}}$ implies $\mathbf{A} \models p \simeq q$ for each $\mathbf{A}$.

Furthermore, an environment $\vec{\rho}$ is said to be *admissible* iff its image can generate a perfect sub-eBA, i.e. $[\vec{\rho}(V \cup F)]$ is a perfect sub-eBA. Then, we can define the above concepts accordingly in the context of the admissibility. Formally,

Definition 7.1 (*Admissible satisfaction* $\dot{\models}_{\mathrm{eBA}}$). For ordinary term $p, q \in T$ and function terms (with arity m)$p, q \in FT_m (m \in Nat)$, we define the following:

(i) $\mathbf{A} \dot{\models}_{\mathrm{eBA}} p \simeq q$ (or $\mathbf{A} \dot{\models} p \simeq q$) iff $\mathscr{A}[\![p]\!](\vec{\rho}) = \mathscr{A}[\![q]\!](\vec{\rho})$ for every admissible environment $\vec{\rho}$ on $\mathbf{A}$;

(ii) $\mathscr{K} \dot{\models}_{\mathrm{eBA}} p \simeq q$ (or $\mathscr{K} \dot{\models} p \simeq q$) iff $\mathbf{A} \dot{\models} p \simeq q$ for each eBA $\mathbf{A} \in \mathscr{K}$;

(iii) $\mathscr{K} \dot{\models}_{\mathrm{eBA}} \Gamma$ (or $\mathscr{K} \dot{\models} \Gamma$) iff $\mathscr{K} \dot{\models} p \simeq q$ for each $p \simeq q \in \Gamma$;

(iv) $Adm_{\Sigma}(\Gamma)$ (or simply $Adm(\Gamma)$ when Σ can be decided by context since the (binding) signature Σ is assumed to be arbitrarily fixed) is defined to be $\{\mathbf{A} \mid \mathbf{A} \dot{\models} \Gamma\}$;

(v) $\Gamma \dot{\models}_{\mathrm{eBA}} p \simeq q$ iff for each eBA $\mathbf{A}$, $\mathbf{A} \dot{\models} \Gamma$ implies $\mathbf{A} \dot{\models} p \simeq q$;

(vi) $\dot{I}_{\mathscr{K}}(\mathbf{X}) =_{\mathrm{df}} \{p \simeq q \mid \mathscr{K} \dot{\models}_{\mathrm{eBA}} p \simeq q\}$, where $\mathscr{K}$ is a class of eBAs and $\mathbf{X} = \langle V, \{F_k \mid k \in Nat\} \rangle$.

For Definition 7.1, we make the following remark.

(1) if $\mathbf{A} \models p \simeq q$, then $\mathbf{A} \dot{\models} p \simeq q$ where $\mathbf{A}$ is an eBA.

(2) $Mod(\Gamma) \subseteq Adm(\Gamma)$, where Γ is a set of binding equations (BEs).

On the other hand, suppose that Γ^1 and Γ^2 are two sets of BEs, let $p \simeq q \in \Gamma^1$ iff $Adm(\Gamma) \dot{\models} p \simeq q$; and let $p' \simeq q' \in \Gamma^2$ iff $Mod(\Gamma) \models p' \simeq q'$. Then, we know, in general,

(3) neither $p \simeq q \in \Gamma^1$ implies $p \simeq q \in \Gamma^2$

(4) nor $p' \simeq q' \in \Gamma^2$ implies $p' \simeq q' \in \Gamma^1$.

Essentially, these are the relations between the satisfaction $\models_{\mathrm{eBA}}$ and the admissible satisfaction $\dot{\models}_{\mathrm{eBA}}$. More discussions on this will be resumed in Section 9.3.

Lemma 7.2 (Effect of an eBH over an interpretation). *If* $\zeta : \mathbf{A} \to \mathbf{A}'$ *is an eBH, then* $\zeta(\mathscr{A}[\![p]\!](\vec{\rho})) = \mathscr{A}'[\![p]\!](\zeta \circ \vec{\rho})$.

The above lemma implies that the effect of an eBH over an interpretation is completely determined by the effect of the eBH over the environment of the interpretation. By exploiting this result, we have the following.

Lemma 7.3 (Admissible BEs and admissible eBHs). *Let* $\mathscr{K}$ *be a class of eBAs; then,* $\mathscr{K} \dot{\models} p \simeq q$ *iff* $\zeta(\bullet_{[p]}) = \zeta(\bullet_{[q]})$ *for each admissible eBH* $\zeta : \mathbf{T} \to \mathbf{A}$ *and every* $\mathbf{A} \in \mathscr{K}$.

Proof. By Lemma 7.2 and Theorem 4.3. □

Lemma 7.3 implies that the admissible equality can be completely captured by admissible eBHs. Consequently, we have

Theorem 7.4 (Admissible BE and admissible eBC). $\mathscr{K} \dot{\models} p \simeq q$ *iff* $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \dot{\vartheta}_{\mathscr{K}}$, *where* $\mathscr{K}$ *is a class of eBAs.*

Proof. "⇐": We suppose $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \dot{\vartheta}_{\mathscr{K}}$ and given an eBA $\mathbf{A} \in \mathscr{K}$. Let $\zeta : \mathbf{T} \to \mathbf{A}$ be an admissible eBH. Then, we have $\dot{\vartheta}_{\mathscr{K}} \subseteq Ker(\zeta)$. So, there exists an eBH $\check{\zeta} : \dot{\mathbf{T}}_{\mathscr{K}}(\dot{\mathbf{X}}) \to \mathbf{A}$ such that $\zeta = \check{\zeta} \circ v_{\dot{\vartheta}_{\mathscr{K}}}$. Hence,

$$\zeta(\bullet_{[p]}) = \check{\zeta} \circ v_{\dot{\vartheta}_{\mathscr{K}}}(\bullet_{[p]}) = \check{\zeta} \circ v_{\dot{\vartheta}_{\mathscr{K}}}(\bullet_{[q]}) \quad (\text{since } Ker(v_{\mathscr{K}}) = \dot{\vartheta}_{\mathscr{K}}) = \zeta(\bullet_{[q]}).$$

Consequently, $\mathscr{K} \dot{\models} p \simeq q$ (since for every environment $\langle \rho, \varphi \rangle$, there exists an eBH $\zeta : \mathbf{T} \to \mathbf{A}$ such that the environment $\langle \rho, \varphi \rangle$ and the eBH ζ agree on $\mathbf{X}$ or $V \cup F$).

"⇒": We assume $\mathscr{K} \dot{\models} p \simeq q$, i.e. for every admissible eBH $\zeta : \mathbf{T}(\mathbf{X}) \to \mathbf{A}$ where $\mathbf{A} \in \mathscr{K}$, we have $\zeta(\bullet_{[p]}) = \zeta(\bullet_{[q]})$. In other words, $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in Ker(\zeta)$. Therefore, since the previous membership does not depend on each individual eBH ζ, we have $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \bigcap_{\zeta : \mathbf{T} \dot{\to} \mathbf{A}} Ker(\zeta)$. Again, this membership does not depend on each individual eBA $\mathbf{A} \in \mathscr{K}$. So, we have $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \dot{\vartheta}_{\mathscr{K}}$. □

Informally, Theorem 7.4 expresses that the admissible equality is totally decided by a certain eBC. Similar to a Birkhoff's theorem, we have a theorem below. Formally

Theorem 7.5 (Admissible Birkhoff's Theorem). *The following three statements are equivalent*: (1) $\mathscr{K} \dot{\models} p \simeq q$; (2) $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \dot{\vartheta}_{\mathscr{K}}$; (3) $\dot{\mathbf{T}}_{\mathscr{K}}(\dot{\mathbf{X}}) \dot{\models} p \simeq q$.

Proof. The equivalence between (1) and (2) was established in Theorem 7.4. We only consider the equivalence between (2) and (3) below.

(2) ⇒ (3): We have that for every admissible eBH $\zeta : \mathbf{T} \dot{\to} \dot{\mathbf{T}}_{\mathscr{K}}(\dot{\mathbf{X}})$ there exists an eBH $\hat{\zeta} : \dot{\mathbf{T}}_{\mathscr{K}}(\dot{\mathbf{X}}) \to \dot{\mathbf{T}}_{\mathscr{K}}(\dot{\mathbf{X}})$ such that $\zeta = v_{\dot{\vartheta}_{\mathscr{K}}} \circ \hat{\zeta}$. This is because of Theorem 6.5, Corollary 6.7,

and the equivalence between (1) and (2). So, we have that $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \dot{\vartheta}_{\mathscr{K}}$ implies $\dot{\mathbf{T}}_{\mathscr{K}}(\dot{\mathbf{X}}) \models p \simeq q$.

(3) $\Rightarrow$ (2): We have that $\dot{\mathbf{T}}_{\mathscr{K}}(\dot{\mathbf{X}}) \models p \simeq q$ implies $v_{\dot{\vartheta}_{\mathscr{K}}}(\bullet_{[p]}) = v_{\dot{\vartheta}_{\mathscr{K}}}(\bullet_{[q]})$, since the natural eBH $v_{\dot{\vartheta}_{\mathscr{K}}}$ is admissible. Hence, we have $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \dot{\vartheta}_{\mathscr{K}}$. $\square$

The above result shows that the admissibility is indeed a remedy to Birkhoff's approach. Furthermore, it shows that the admissible BEs are captured by admissible eBCs, and this, in turn, can be captured by the term eBA $\mathbf{T}$ through a certain way. The remaining problem is to catch the certain way syntactically. This is the subject of next section.

8. Admissible completeness of the equational logic $\vdash_{\mathrm{eBA}}$

Analogous to the first-order fully invariant congruence [5], we introduce the *admissibly invariant* eBC to characterize the eBC $\dot{\vartheta}_{\mathscr{K}}$ over a class $\mathscr{K}$ of eBAs. To motivate the definition of the admissibly invariant eBCs, we need the following lemma. Namely

Lemma 8.1 (Binding substitutions on $\mathbf{T}$). *Let ϑ be an eBC on the term eBA $\mathbf{T}$. For every eBH $\zeta : \mathbf{T} \to \mathbf{T}/\vartheta$, there exists an eBH $\check{\zeta} : \mathbf{T} \to \mathbf{T}$ such that $\zeta = \check{\zeta} \circ v_\vartheta$, i.e. the diagram in Fig. 4 commutes.*

Proof. For each $x \in X$ and each $f \in X_m$, let us fix a value in $v_\vartheta^{-1}(\zeta(x))$ and in $v_\vartheta^{-1}(\zeta(f))$ for $\check{\zeta}$ on $\mathbf{X}$ by Axiom of Choice. [*Note*: If $\zeta(x) = \zeta(y)$ then $v_\vartheta^{-1}(\zeta(x)) = v_\vartheta^{-1}(\zeta(y))$.] For readability, we keep the notation as $\check{\zeta}(x) =_{\mathrm{df}} v_\vartheta^{-1}(\zeta(x))$ and $\check{\zeta}(f) =_{\mathrm{df}} v_\vartheta^{-1}(\zeta(f))$.

Since $\mathbf{T}$ is the free (term) eBA, $\check{\zeta}$ can be extended to be an eBH from $\mathbf{T}$ to $\mathbf{T}$. Obviously, the restrictions of the eBH ζ and the composition $v_\vartheta \circ \check{\zeta}$ on X are equal, i.e. $\zeta \lceil_{\mathbf{X}} = (v_\vartheta \circ \check{\zeta}) \lceil_{\mathbf{X}}$. By the uniqueness of the free eBA on eBHs, we come to $\zeta = v_\vartheta \circ \check{\zeta}$. $\square$

As you may notice, the eBH $\check{\zeta}$ is actually a *substitution* as it is commonly called. This observation leads us to the definition of the admissibly invariant eBCs. Formally,

Definition 8.2 (*Admissibly invariant eBC*). An eBC ϑ on the term eBA $\mathbf{T}$ is admissibly invariant iff for every admissible eBH $\zeta : \mathbf{T} \to \mathbf{T}/\vartheta$, we have that $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \vartheta$ implies $\langle \check{\zeta}(\bullet_{[p]}), \check{\zeta}(\bullet_{[q]}) \rangle \in \vartheta$

Some obvious examples of the admissibly invariant eBCs are

(a) $\mathbf{T} \times \mathbf{T}$ is an admissibly invariant eBC on $\mathbf{T}$;

(b) $\dot{\vartheta}_{\mathscr{K}}$ (Definition 6.11) is an admissibly invariant eBC on $\mathbf{T}$ for a class $\mathscr{K}$ of eBAs.

To justify the introduction of the admissibly invariant concept, we provide a lemma below. Formally

$$\begin{tikzcd} \mathbf{T} \arrow[r, "\check{\zeta}"] \arrow[dr, "\zeta"'] & \mathbf{T} \arrow[d, "\nu_\vartheta"] \\ & \mathbf{T}/\vartheta \end{tikzcd}$$

Fig. 4. Binding substitution.

Lemma 8.3 (Admissibly invariant eBC and admissible BE). *Let ϑ be an admissibly invariant eBC on the term eBA* $\mathbf{T}$. *Then,* $\mathbf{T}/\vartheta \models p \simeq q$ *iff* $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \vartheta$.

Proof. Let $\{\vec{f}, \vec{x}\} = FreeVar(p) \cup FreeVar(q)$. So, we consider the environments over $\{\vec{f}, \vec{x}\}$.

"$\Leftarrow$": $\mathbf{T}/\vartheta \models p \simeq q \Leftrightarrow$ for all admissible $\langle \rho, \varphi \rangle$, $\mathscr{T}[\![p]\!](\rho, \varphi) = \mathscr{T}[\![q]\!](\rho, \varphi)$.

From these $\mathscr{T}$ and $\langle \rho, \varphi \rangle$, we can get an admissible eBH $\zeta : \mathbf{T} \to \mathbf{T}/\vartheta$ such that $\mathscr{T}[\![p]\!](\rho, \varphi) = \mathscr{T}[\![q]\!](\rho, \varphi) \Leftrightarrow \zeta(\bullet_{[p]}) = \zeta(\bullet_{[q]})$.

Conversely, given an admissible eBH $\zeta : \mathbf{T} \to \mathbf{T}/\vartheta$, we have an admissible environment $\langle \rho, \phi \rangle$ such that $\mathscr{T}[\![p]\!](\rho, \varphi) = \mathscr{T}[\![q]\!](\rho, \varphi) \Leftrightarrow \zeta(\bullet_{[p]}) = \zeta(\bullet_{[q]})$.

Since $\nu_\vartheta : \mathbf{T} \to \mathbf{T}/\vartheta$ (the natural eBH associated with the eBC ϑ) is onto, we have that $\nu_\vartheta(\check{\zeta}(\bullet_{[p]})) = \nu_\vartheta(\check{\zeta}(\bullet_{[q]}))$, i.e. $\langle \check{\zeta}(\bullet_{[p]}), \check{\zeta}(\bullet_{[q]}) \rangle \in \vartheta$. Because of the admissibly invariant property of the eBC ϑ, we have that $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \vartheta$ implies for every eBH $\check{\zeta} : \mathbf{T} \to \mathbf{T}, \langle \check{\zeta}(\bullet_{[p]}), \check{\zeta}(\bullet_{[q]}) \rangle \in \vartheta$.

"$\Rightarrow$": Since ν_ϑ (the natural eBH associated with the eBC ϑ) is admissible, we have that $\mathbf{T}/\vartheta \models p \simeq q$ implies $\nu_\vartheta(\bullet_{[p]}) = \nu_\vartheta(\bullet_{[q]})$. In other words, we have $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \vartheta$. $\square$

The above lemma says that an admissible invariant eBC decides the admissible equality of its quotient eBA. This convinces us that we are on the right track.

Definition 8.4 (*Least admissible invariance containing* $\vec{E}$). Let $\vec{E}$ be a family of both $E \subseteq T \times T$ and $E_k \subseteq FT_k \times FT_k$ for $k \in Nat$. Then, $\vartheta_{Adm}(\vec{E})$ denote the *least admissibly invariant* eBC on the term eBA $\mathbf{T}$ containing $\vec{E}$.

Sometimes, the eBC $\vartheta_{Adm}(\vec{E})$ is called the *admissibly invariant eBC generated by* $\vec{E}$. This comes from the fact that $\vartheta_{Adm}(\vec{E}) =_{\mathrm{df}} \bigcap \Psi(\vec{E})$ where $\Psi(\vec{E})$ is the collection of all possible admissibly invariant eBCs on the term eBA $\mathbf{T}$ containing $\vec{E}$. Such a definition is well-defined, since $\mathbf{T} \times \mathbf{T}$ is an admissibly invariant eBC (i.e. $\Psi(\vec{E}) \neq \emptyset$) and the property of the admissibly invariant is preserved by an arbitrary intersection (e.g. $\dot{\vartheta}_1 \cap \dot{\vartheta}_2$ is admissibly invariant if both of them are).

Definition 8.5 (*Function* ϕ). Given a pair $\langle X, \{X_k \mid k \in Nat\}\rangle$ where X is set of ordinary variables and X_k is a set of function variables with arity $k \in Nat$. Let $\mathbf{X}$ be such a pair, and the map $\phi : \langle\langle \dot{I}(\mathbf{X})\rangle\rangle \to \mathbf{T}\times\mathbf{T}$ be the bijection defined by $\phi([p]\simeq[q]) =_{\mathrm{df}} \langle \bullet_{[p]}, \bullet_{[q]}\rangle$, where $\langle\langle \dot{I}(\mathbf{X})\rangle\rangle =_{\mathrm{df}} \{[p]\simeq[q] \mid p\simeq q \in \dot{I}(\vec{X})\}$ $[p] =_{\mathrm{df}} \{q \mid p \approx q\}$, and $\dot{I}(\mathbf{X})$ is a set of BEs.

Obviously, we have $\vartheta_{Adm}(\phi(\Gamma)) = \dot{\vartheta}_{Adm(\Gamma)}$ for every $\Gamma \subseteq (T\times\{\simeq\}\times T)\cup(FT\times\{\simeq\}\times FT)$. (Hint: see the obvious examples of previously given admissibly invariant eBCs and the least closure condition of the eBC ϑ_{Adm} for one direction of the inclusions, and see Lemma 8.3 to get the fact of the quotient eBA $\mathbf{T}/\vartheta_{Adm}(\phi(\Gamma)) \in Adm(\Gamma)$ for the other direction of the inclusions.) We state it as theorem. Formally,

Theorem 8.6 (Least admissible invariance and admissible BEs). *$\Gamma \dot{\models} p\simeq q$ iff $\langle \bullet_{[p]}, \bullet_{[q]}\rangle \in \vartheta_{Adm}(\phi(\Gamma))$.*

Such an observation leads to a concept of *admissible substitutions*. Formally

Definition 8.7 (*Admissible substitution with respect to* Γ). Given a set Γ of (admissible) BEs, a substitution $\vec{\varrho}$ is said to be *admissible* with respect to Γ iff the corresponding eBH $\zeta : \mathbf{T}\to\mathbf{T}$ (i.e. $\zeta(\bullet_{[p]}) = \bullet_{[p\vec{\varrho}]}$) has the property that the composition $\nu_{\vartheta_{Adm}(\phi(\Gamma))}\circ\zeta$ is an admissible eBH.

We denote an admissible substitution (map) $\vec{\varrho}$ as $\vec{\varrho} \in Sub_{Adm}(\Gamma)$. An example of such a substitution (map) is the *identity* substitution (map) $\vec{\imath} \in Sub_{Adm}(\Gamma)$.

The well-definedness of Definition 8.7 is guaranteed by the fact that $\mathbf{T}$ is a free (term) eBA over all possible eBAs. From this, we can easily get a lemma. Namely

Lemma 8.8 (Admissible BEs and admissible substitutions). *If $Adm(\Gamma) \dot{\models} p\simeq q$ then $Adm(\Gamma) \dot{\models} p\vec{\varrho} \simeq q\vec{\varrho}$, where $\vec{\varrho} \in Sub_{Adm}(\Gamma)$.*

Proof. We simply have a corresponding eBH ζ for $\vec{\varrho}$. Then, since the composition $\nu_{\vartheta_{Adm}(\phi(\Gamma))}\circ\zeta$ is admissible and $\vartheta_{Adm}(\phi(\Gamma)) = \dot{\vartheta}_{Adm(\Gamma)}$, we have that $\langle \bullet_{[p]}, \bullet_{[q]}\rangle \in \dot{\vartheta}_{Adm(\Gamma)}$ implies $\langle \zeta(\bullet_{[p]}), \zeta(\bullet_{[q]})\rangle \in \dot{\vartheta}_{Adm(\Gamma)}$. Putting it in another way, we have that $Adm(\Gamma) \dot{\models} p\simeq q$ implies $Adm(\Gamma) \dot{\models} p\vec{\varrho} \simeq q\vec{\varrho}$. $\square$

From Lemma 8.8, we understand that the eBC $\dot{\vartheta}_{Adm(\Gamma)}$ is closed under the admissible substitutions. However, we have no *feasible* criteria to judge whether a given substitution is admissible. Nevertheless, the effect of applying substitutions to binding terms is completely decided by the images of the substitutions on both the free ordinary variables and free function variables of the binding terms. Luckily, there is a way to get around the *explicit* feasible criteria. That is, (a) the number of free ordinary variables and free function variables in each binding term involved must be *finite*, and (b) the identity substitution $\vec{\imath}$ is obviously admissible. So, by shifting the variables in the identity substitution $\vec{\imath}$ both to preserve the admissibility and to share the images of an arbitrary substitution $\vec{\varrho}$ on the free variables of a binding term p, we can get an admissible substitution $\vec{\varrho}$ such that $p\vec{\varrho} = p\vec{\varrho}$. Hence, we can use any *arbitrary* substitution in practice. This completes our quest for a syntactic characterization of the admissible equality. To confirm this claim, we define the admissible equational logic $\vdash_{\mathrm{eBA}}$ (or simply $\vdash$) as below and prove the claim in Theorem 8.10.

The admissible equational logic $\vdash$ is almost the same as the first-order many-sorted equational logic $\vdash_{\mathrm{EQ}}$ in [28, 11] (or [29, Ch. 2]) except that (a) it has three extra rules (α), (ξ) and (ξ^{-1}); (b) composition rule (eq-cmp) in $\vdash_{\mathrm{EQ}}$ is replaced by two rules (cmp-1) and (cmp-2); (c) its substitution is a family $\vec{\varrho}$ of mappings instead of a single mapping $\imath$; (d) meta variables are mainly binding terms p, q, r (which can be either ordinary terms or function terms) instead of first-order ordinary terms t, u, v respectively.

Definition 8.9 (*Admissible equational logic* $\vdash_{\mathrm{eBA}}$). The admissible equational logic $\vdash_{\mathrm{eBA}}$ (or simply $\vdash$) is the same as defined in Section 7.

Let $\dot{\mathscr{M}}_{\mathrm{BE}}$ (or simply $\dot{\mathscr{M}}$) be a function which produces a set $\dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)$ when given a set Γ of BEs, i.e. $p \simeq q \in \dot{\mathscr{M}}(\Gamma)$ iff $\Gamma \vdash_{\mathrm{eBA}} p \simeq q$ without using the *cut* rules. Obviously, $\dot{\mathscr{M}}_{\mathrm{BE}}$ is monotonic, and we can easily have that $p \simeq q \in \bigsqcup \dot{\mathscr{M}}(\Gamma)$ iff $\Gamma \vdash_{\mathrm{eBA}} p \simeq q$.

Theorem 8.10. $\phi(\langle\langle\bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)\rangle\rangle) = \vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle))$.

Proof. (i) First of all, we define a new rule (adm-ξ), called the admissible ξ rule, as

$$(\text{adm-}\xi)\quad \frac{\{\Gamma \vdash t\,[\vec{x} := \vec{u}] \simeq t'\,[\vec{x} := \vec{u}] \mid u_j \in T\,(j = 1, 2, \ldots)\}}{\Gamma \vdash \langle \vec{x} : t\rangle \simeq \langle \vec{x} : t'\rangle}$$

where $\vec{\imath}[\vec{x} := \vec{u}]$ is a family of substitution functions and $\{\vec{x}\} \subseteq V$.

Then, we understand that if we define a new $\dot{\mathscr{M}}_{\mathrm{BE}}$ which is alomst the same as the previous one except that this new $\dot{\mathscr{M}}$ is without the (ξ) rule but with the new (adm-ξ) rule, both of the old and new $\dot{\mathscr{M}}$s are actually equivalent to each other. (Hint: comparing the two premises of (ξ) and (adm-ξ) rules with both (b-sub) rule and the fact that the identity substitution $\vec{\imath}$ is admissible for arbitrary Γ.)

(ii.a) $\phi(\langle\langle\bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)\rangle\rangle)$ is obviously an equivalence relation containing $\phi(\langle\langle\Gamma\rangle\rangle)$ (for the new $\dot{\mathscr{M}}$).

(ii.b) By (α) rule, (ξ^{-1}) rule, (b-sub) rule and the new (adm-ξ) rule with the above (ii.a), we have $\phi(\langle\langle\bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)\rangle\rangle)$ satisfies (eBH-ext).

(ii.c) By both (cmp-1) and (cmp-2) rules with the above (ii.a) and (ii.b), we have that $\phi(\langle\langle\bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)\rangle\rangle)$ is an eBC.

(ii.d) By the (b-sub) rule, we get that $\phi(\langle\langle\bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)\rangle\rangle)$ is admissibly invariant.

So, combining all items in (i) and (ii), we know

$$\phi(\langle\langle\bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)\rangle\rangle) \supseteq \vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle)).$$

(iii) On the other hand, it is quite easy to see that $\phi^{-1}(\vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle)))$ is closed under the following rules, where $\phi^{-1}(\vartheta) =_{\mathrm{df}} \{[p] \simeq [q] \mid \langle \bullet_{[p]}, \bullet_{[q]}\rangle \in \vartheta\}$ (and $[p] = \{q \mid p \dot{\approx} q\}$):

(iii.a) for (α) rule, trivial;

(iii.b) for (ξ^{-1}) rule, a simple exercise (hint: $\bullet_{[\langle \vec{z} : u\rangle]}(\bullet_{[\vec{y}\,]}) = \bullet_{[u\,[\vec{z} := \vec{y}\,]]}$);

(iii.c) for (cmp-1) and (cmp-2) rules, trivial;

(iii.d) for (b-sub) rule, an easy exercise (hint: if $\check{\zeta} \in Sub_{Adm}(\Gamma)$, then the composition $v_{\vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle))} \circ \check{\zeta}$ is an admissible eBH);

(iii.e) for (adm-ξ) rule, we show the closeness below: suppose $\langle \bullet_{[t]}, \bullet_{[t']}\rangle \in \vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle))$.

For every (substitution) eBH $\zeta : \mathbf{T} \to \mathbf{T}$ such that the composition $\check{\zeta} \circ v_{\vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle))}$ is an admissible eBH. Let $\zeta(x_j)$ be u_j ($1 \leqslant j \leqslant |\vec{x}|$), we have that $\bullet_{[\langle \vec{x} : t\rangle]}(\bullet_{[u_1]}, \bullet_{[u_2]}, \ldots, \bullet_{[u_{|\vec{x}|}]}) = \bullet_{[t\,[\vec{x} := \vec{u}\,]]} = \zeta(t)$ and $\bullet_{[\langle \vec{x} : t'\rangle]}(\bullet_{[u_1]}, \bullet_{[u_2]}, \ldots, \bullet_{[u_{|\vec{x}|}]}) = \bullet_{[t'\,[\vec{x} := \vec{u}\,]]} = \zeta(t')$. Hence, we have $\langle \zeta(t), \zeta(t')\rangle \in \vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle))$, since the composition $\check{\zeta} \circ v_{\vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle))}$ is an admissible eBH

Then, by (eBH-ext), we get that $\langle \bullet_{[\langle \vec{x} : t\rangle]}, \bullet_{[\langle \vec{x} : t'\rangle]}\rangle \in \vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle))$. In other words, $\phi^{-1}(\vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle)))$ is closed under the admissible (adm-ξ) rule.

Therefore, by both (i) and the least closure condition of $\bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)$, we have

$$\phi\left(\left\langle\left\langle \bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)\right\rangle\right\rangle\right) \supseteq \vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle)).$$

(iv) By (ii) and (iii) above, we come to $\phi(\langle\langle \bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)\rangle\rangle) = \vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle))$. □

From Theorem 8.10, we can infer that $\Gamma \dot{\models} p \simeq q$ iff $p \simeq q \in \bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)$, since $\Gamma \dot{\models} p \simeq q$ iff $\dot{\vartheta}_{Adm_{\Sigma}(\Gamma)}$ iff $\langle \bullet_{[p]}, \bullet_{[q]}\rangle \in \vartheta_{Adm}(\phi(\langle\langle\Gamma\rangle\rangle))$ iff $\langle \bullet_{[p]}, \bullet_{[q]}\rangle \in \phi(\langle\langle \bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)\rangle\rangle)$ iff $p \simeq q \in \bigsqcup \dot{\mathscr{M}}_{\mathrm{BE}}(\Gamma)$. So, I state these observations as a corollary.

Corollary 8.11 (Soundness and completeness of $\dot{\vdash}_{\mathrm{eBA}}$). *$\Gamma \dot{\models} p \simeq q$ iff $\Gamma \dot{\vdash} p \simeq q$.*

Although we have no general syntactic characterization of the admissible substitutions, this is hardly a limitation (see the comments before Definition 8.9). Of course, a general characterization of the admissible substitutions (and/or non-admissible substitutions) deserves our future attention.

9. Discussions

9.1. Admissible variety problem

Since the admissible models here seems not to be closed under *products*, there is an open problem whether there is a Birkhoff-like *variety* concept for eBAs such that it is the least closure of certain operations. On the other hand, we know that the quotient (term) eBA $\dot{\mathbf{T}}(\dot{\mathbf{X}})$ admissibly satisfies $\dot{I}_{\mathscr{K}}$ (Definition 7.1 see (vi)) for the class $\mathscr{K}$ of eBAs. This naturally leads to a question whether there is an operation to replace the products of eBAs, which enables us to get a Birkhoff's variety concept.

Definition 9.1.1 (*Direct product of eBAs*). Let $\mathbf{A}_k$ $(k \in Ind)$ be an indexed family of eBAs, the *direct product* $\mathbf{A}$, written as $\mathbf{A} = \prod_{k \in Ind} \mathbf{A}_k$, is defined as

(a) $g^{\mathbf{A}} = \langle g^{\mathbf{A}_k} \rangle_{k \in Ind}$ and $g^{\mathbf{A}}(\vec{a})(k) = g^{\mathbf{A}_k}(\vec{a}_k)$, where $(a_k)_j \in A_k$ $(1 \leqslant j \leqslant |\vec{a}_k|)$ and $k \in Ind$.

(b) $\sigma^{\mathbf{A}}(\vec{g}^{\mathbf{A}}, \vec{a})(k) = \sigma^{\mathbf{A}_k}(g_1^{\mathbf{A}_k}, g_2^{\mathbf{A}_k}, \ldots, g_{|\vec{m}|}^{\mathbf{A}_k}, \vec{a}_k)$ for $k \in Ind$ and $g_i^{\mathbf{A}_k} \in \mathscr{F}_{m_i}^{\mathbf{A}_k}$ (the domain of $\mathscr{F}_{m_i}^{\mathbf{A}_k}$ is A_k) and $(a_k)_j \in A_k$.

Apparently, eBAs are closed under direct products. For sub-direct products, we introduce them as follows.

Definition 9.1.2 (*Sub-direct product*). An sub-eBA $\mathbf{A}$ $(= \sum_{k \in Ind} A_k)$ of $\prod_{k \in Ind} \mathbf{A}_k$, where $\mathbf{A}_k$ is an eBA indexed by $k \in Ind$, is said to be a *sub-direct product* of the indexed family $\mathbf{A}_k$ $(k \in Ind)$ of eBAs iff

(i) $\mathbf{A}$ is a perfect sub-eBA of $\prod_{k \in Ind} \mathbf{A}_k$;

(ii) $\pi_k(\mathbf{A}) = \mathbf{A}_k$ for each $k \in Ind$.

Similar to first-order algebras, we introduce sub-direct embedding as follows.

Definition 9.1.3 (*Sub-direct embedding*). An embedding (which means a 1–1 eBH) $\zeta : \mathbf{A} \to \prod_{k \in Ind} \mathbf{A}_k$ is a *sub-direct embedding* iff the image $\zeta(\mathbf{A})$ is a sub-direct product of the family $\mathbf{A}_k$ $(k \in Ind)$.

An example of the sub-direct embedding is given below.

Lemma 9.1.4. (Natural eBH and sub-direct embedding). *If ϑ_k is an eBC on an eBA $\mathbf{A}$, and $\bigcap_{k \in Ind} \vartheta_k = \mathbf{Diag}$ (diagonal relation, it is an eBC), then the natural eBH $v : \mathbf{A} \to \prod_{k \in Ind}(\mathbf{A}/\vartheta_k)$ defined by $v(g)(k) = g/\vartheta_k$ for $g \in \mathscr{F}_m^{\mathbf{A}}(A)$ $(m \geqslant 0)$ is a sub-direct embedding.*

The intersection of a collection of eBCs is an eBC, and it has an interesting property with the members of the collection. The property is formalized as follows.

Lemma 9.1.5 (Intersection of eBCs). *Let $\vartheta = \bigcap_{k \in Ind} \vartheta_k$ and ϑ_k be an eBC on an eBA $\mathbf{A}$. Then, the intersection $\bigcap_{k \in Ind}(\vartheta_k/\vartheta)$ is the eBC* **diag** *on the quotient eBA $\mathbf{A}/\vartheta$, where* **diag** *is the obvious diagonal eBC.*

Also, the isomorphic property among components of a product of eBAs is preserved by the product. Formally,

Lemma 9.1.6 (Isomorphism and product). *If each eBA $\mathbf{A}_k$ is isomorphic to $\mathbf{A}'_k$ $(k \in Ind)$, then the product $\prod_{k \in Ind} \mathbf{A}_k$ is isomorphic to the product $\prod_{k \in ind} \mathbf{A}'_k$.*

There is an simple fact about preservation of perfectness under eBHs. That is,

$$\begin{tikzcd} \mathbf{A}/\vartheta \arrow[r, "\nu"] & \prod_{k\in Ind}(\mathbf{A}/\vartheta)/(\vartheta_k/\vartheta) \arrow[d, "\zeta"] \\ & \prod_{k\in Ind}\mathbf{A}/\vartheta_k \end{tikzcd}$$

Fig. 5. Quotient eBA and sub-direct embedding.

Lemma 9.1.7 (Perfectness and eBHs). *If* $\mathbf{A}$ *is a perfect sub-eBA of an eBA* $\mathbf{A}'$, *and the eBH* $\zeta:\mathbf{A}'\to\mathbf{A}''$ *is* 1–1, *then the image* $\zeta(\mathbf{A})$ *is a perfect sub-eBA of* $\mathbf{A}''$.

This lemma can be regarded as an improvement of Lemma 6.3.

For a collection of eBCs and their quotient eBAs, the following is an interesting fact.

Theorem 9.1.8 (Quotient eBA and sub-direct embedding). *If* $\mathbf{A}$ *is an eBA and* ϑ_k *is an eBC* ($k\in Ind$), *let* $\vartheta=\bigcap_{k\in Ind}\vartheta_k$; *then the quotient eBA* $\mathbf{A}/\vartheta$ *can be sub-directly embedded in the product* $\prod_{k\in Ind}\mathbf{A}/\vartheta_k$.

Proof. See Fig. 5. By Lemmas 9.1.4, 9.1.5 and 9.1.7, the quotient eBA $\mathbf{A}/\vartheta$ can be sub-directly embedded in the product $\prod_{k\in Ind}(\mathbf{A}/\vartheta)/(\vartheta_k/\vartheta)$. By Lemmas 5.11 and 9.1.6, we have that the product $\prod_{k\in Ind}(\mathbf{A}/\vartheta)/(\vartheta_k/\vartheta)$ is isomorphic to the product $\prod_{k\in Ind}\mathbf{A}/\vartheta_k$. □

Theorem 9.1.9 (Admissible quotient eBA). *For a class* $\mathscr{K}\neq\emptyset$, *we have the admissible quotient eBA* $\dot{\mathbf{T}}_{\mathscr{K}}(\dot{\mathbf{X}})\in\mathbf{IS}_p\mathbf{P}(\mathscr{K})$; *in particular, if* $\mathscr{K}$ *is closed under* $\mathbf{I}$, $\mathbf{S}_p$ *and* $\mathbf{P}$, *then* $\dot{\mathbf{T}}_{\mathscr{K}}(\dot{\mathbf{X}})\in\mathscr{K}$, *where*

$$\mathbf{I}(\mathscr{K})=\{\mathbf{A}'\mid \text{for some } \mathbf{A}\in\mathscr{K} \text{ and } \mathbf{A}'\cong\mathbf{A}\}\ (\cong \text{ means “is isomorphic to”}),$$

$$\mathbf{S}_p(\mathscr{K})=\{\mathbf{A}'\mid \text{for some } \mathbf{A}\in\mathscr{K} \text{ and } \mathbf{A}'\preccurlyeq\mathbf{A}\},$$

$$\mathbf{P}(\mathscr{K})=\{\mathbf{A}'\mid \mathbf{A}' \textit{ is a product of a subset of } \mathscr{K}\}.$$

There is an interesting question which comes from Theorem 9.1.9. Namely,

Question 9.1.10 (Closure under products). *Under what condition, especially a equationally definable condition,* $\mathscr{K}$ *is closed under direct product?*

Also, we name the existence of the operator under which a Birkhoff-like (binding) variety is closed as an open problem.

Open Problem 9.1.11 (Admissible variety). *Whether there is an constructible operator Υ over a class $\mathscr{K}$ of eBAs such that $Adm(\dot{I}_{\mathscr{K}}) = \Upsilon(\mathscr{K})$?*

9.2. Logical relations vs. admissibility

In this section, we will discuss the relationship between Plotkin's logical relations and the admissibility.

Definition 9.2.1 (*Logical relation on eBAs*). A relation $\zeta \subseteq \mathbf{A} \times \mathbf{A}'$ is *logical* iff given $g \in \mathscr{F}_m^{\mathbf{A}}$ and $h \in \mathscr{F}_m^{\mathbf{A}'}$, $\langle g, h\rangle \in \zeta_m \Leftrightarrow (\forall a_j \in A, \forall b_j \in A'. \bigwedge_{j=1}^{m} \langle a_j, b_j\rangle \in \zeta \Rightarrow \langle g(\vec{a}), h(\vec{b})\rangle \in \zeta$.

Let $\zeta : \mathbf{A} \to \mathbf{A}'$ be an eBH; then we have that it is logical iff for all $m \geqslant 0$, $g \in \mathscr{F}_m^{\mathbf{A}}$ and $h \in \mathscr{F}_m^{\mathbf{A}'}$ such that $\forall a_j \in A. \zeta(g(\vec{a})) = h(\zeta(\vec{a})) \Rightarrow \zeta(g) = h$.

From this, we understand that the image of a logical eBH ζ has enough points to identify not only (a) the functions in its image function spaces [i.e. $\zeta(\mathscr{F}^{\mathbf{A}'})$] but also (b) all functions in the function spaces of $\mathscr{F}^{\mathbf{A}'}$.

In contrast, the image of an admissible eBH ζ can only identify the functions in its image function spaces (i.e. only (a) holds). Consequently,

Lemma 9.2.2 (Logical and admissible). *Let $\zeta : \mathbf{A} \to \mathbf{A}'$ be an eBH; then, that ζ is logical implies that ζ is admissible*

This lemma implies that an eBH is certainly admissible if it is logical.

9.3. Completeness and admissible completeness

In this section, we will discuss more on relationship between the binding satisfaction $\models_{\mathrm{eBA}}$ and the admissible binding satisfaction $\dot{\models}_{\mathrm{eBA}}$ (or between $Mod(\Gamma)$ and $Adm(\Gamma)$ for given Γ). It intends to clarify more on the relationship between the completeness and the admissible completeness of the equational logic $\dot{\vdash}_{\mathrm{eBA}}$.

Let us start with summarizing the main results of Birkhoff's method (see [28] or [29, Section 2.1] for instance) and compare these with the ones in this paper.

From first-order algebras [5, 6, 8, 12, 11, 28], we can get a congruence θ_Γ by given Γ through (first-order) many-sorted equational logic $\vdash_{\mathrm{EQ}}$, i.e.

(9.3.a) $\langle X, t, u\rangle \in \theta_\Gamma$ iff $\Gamma \vdash_{\mathrm{EQ}} t \simeq_X u$ (or usually $\Gamma \vdash_{\mathrm{EQ}} \forall X. t \simeq u$ where $t, u \in \mathbf{T}_{\Sigma'}(X)$).

This θ_Γ is cross-invariant so that (9.3.b) $\mathbf{T}/\theta_\Gamma \models_{\mathrm{EQ}} t \simeq_X u$ iff $\langle X, t, u\rangle \in \theta_\Gamma$.

Therefore, we have that (9.3.c) $\Gamma \vdash_{\mathrm{EQ}} t \simeq_X u$ iff $\mathbf{T}/\theta_\Gamma \models_{\mathrm{EQ}} t \simeq_X u$.

Furthermore, by Birkhoff theorems (through the freeness), there is another congruence ϑ such that

(9.3.d) $\Gamma \models_{\mathrm{EQ}} t \simeq_X u$ iff $\mathbf{T}/\vartheta \models_{\mathrm{EQ}} t \simeq_X u$. Then, by establishing the coincidence between ϑ and θ_Γ, we have the first order Completeness of $\vdash_{\mathrm{EQ}}$; i.e. (9.3.e) $\Gamma \models_{\mathrm{EQ}} t \simeq_X u$ iff $\Gamma \vdash_{\mathrm{EQ}} t \simeq_X u$.

Similarly, we can carry the same argument about $\vdash_{\mathrm{EQ}}$ to the present $\dot{\vdash}_{\mathrm{eBA}}$; i.e. (9.3.a') $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \vartheta_\Gamma$ iff $\Gamma \dot{\vdash}_{\mathrm{eBA}}\ p \simeq q$.

We can get that (9.3.b') $\mathbf{T}/\vartheta_\Gamma \models_{\mathrm{eBA}}\ p \simeq q$ iff $\langle \bullet_{[p]}, \bullet_{[q]} \rangle \in \vartheta_\Gamma$.

Therefore, we have that (9.3.c′) $\Gamma \dot{\vdash}_{\mathrm{eBA}}\ p \simeq q$ iff $\mathbf{T}/\vartheta_\Gamma \models_{\mathrm{eBA}}\ p \simeq q$.

However, the crucial point of this paper is that the validation (9.3.d′) cannot be established by Birkhoff method (through the eBA's freeness): (9.3.d′) $\Gamma \models_{\mathrm{eBA}}\ p \simeq q$ iff $\mathbf{T}/\vartheta_\Gamma \models_{\mathrm{eBA}}\ p \simeq q$. That is, we do not know whether (9.3.d′) $Mod(\Gamma) \models_{\mathrm{eBA}}\ p \simeq q$ iff $\mathbf{T}/\vartheta_\Gamma \models_{\mathrm{eBA}}\ p \simeq q$.

The reason for this is that the quotient eBA $\mathbf{T}/\vartheta_\Gamma$ cannot be established as a free eBA over the class $Mod(\Gamma)$ of eBAs. We are only able to establish that $\mathbf{T}/\vartheta_\Gamma$ is admissibly free over an enlarged class $Adm(\Gamma)$, where $Mod(\Gamma) \subseteq Adm(\Gamma)$ since $\mathbf{A} \models_{\mathrm{eBA}} p \simeq q$ implies $\mathbf{A} \dot{\models}_{\mathrm{eBA}} p \simeq q$ for any eBA $\mathbf{A}$; i.e.

(9.3.d$*$) $Adm(\Gamma) \dot{\models}_{\mathrm{eBA}} p \simeq q$ iff $\mathbf{T}/\vartheta_\Gamma \dot{\models}_{\mathrm{eBA}} p \simeq q$.

Actually, it is easy to verify that (9.3.f) $\mathbf{T}/\vartheta_\Gamma \dot{\models}_{\mathrm{eBA}} p \simeq q$ iff $\mathbf{T}/\vartheta_\Gamma \models_{\mathrm{eBA}} p \simeq q$. So, we have that (9.3.e′) $\Gamma \dot{\models}_{\mathrm{eBA}} p \simeq q$ iff $\Gamma \dot{\vdash}_{\mathrm{eBA}}\ p \simeq q$.

Secondly, we know that (9.3.g) $Adm(\Gamma) \models_{\mathrm{eBA}}\ p \simeq q$ implies both $Adm(\Gamma) \dot{\models}_{\mathrm{eBA}} p \simeq q$ and $Mod(\Gamma) \models_{\mathrm{eBA}}\ p \simeq q$; and (9.3.h) either $Adm(\Gamma) \dot{\models}_{\mathrm{eBA}} p \simeq q$ or $Mod(\Gamma) \models_{\mathrm{eBA}}\ p \simeq q$ implies $Mod(\Gamma) \dot{\models}_{\mathrm{eBA}} p \simeq q$.

Question 9.3.1. *What is the relation between $Mod(\Gamma) \models_{\mathrm{eBA}}\ p \simeq q$ and $Adm(\Gamma) \dot{\models}_{\mathrm{eBA}}\ p \simeq q$?*

We know that $Mod(\Gamma) \subseteq Adm(\Gamma)$. So, if $Mod(\Gamma) \supseteq Adm(\Gamma)$, then we would have the coincidence between the completeness and the admissible completeness. Unfortunately, we are not able to establish this. In what follows, we are trying to identify where it is possible to cause a difference between $Adm(\Gamma)$ and $Mod(\Gamma)$.

Recall (9.3.f), we know that the crucial point is that there exists a surjective eBH from $\mathbf{T}$ to $\mathbf{T}/\vartheta_\Gamma$. This can be further generalized to the following.

Lemma 9.3.2. *Let $\mathbf{A} = \langle \langle A, \mathscr{F} \rangle, \mathscr{A} \rangle$ be an eBA such that $|A| \leqslant \aleph_0$ (i.e. up to the countable infinite cardinal), then $\mathbf{A} \models_{\mathrm{eBA}} p \simeq q$ iff $\mathbf{A} \dot{\models}_{\mathrm{eBA}} p \simeq q$.*

From this lemma, we know that $\mathbf{A} \in Adm(\Gamma)$ and $|A| \leqslant \aleph_0$ implies $\mathbf{A} \in Mod(\Gamma)$. For the larger cardinals, i.e. $|A| > \aleph_0$, we can think of applying the idea of the existence of surjective eBHs from a term eBA to $\mathbf{A}$ here. In other words, we can enlarge the set V to V' such that $|V'| = \aleph \geqslant \aleph_0 = |V|$ and obtain a new term eBA $\mathbf{T}(V', F)$ instead of the old term eBA $\mathbf{T}(V, F)$ (see [31, 16, 13] for references on large cardinals). We shall abbreviate these two (old and new) term eBAs as $\mathbf{T}'$ and $\mathbf{T}$, respectively. Furthermore, we introduce the following:

(i) $\mathbf{A} \models^{\aleph}_{\mathrm{eBA}} p \simeq q$ iff $\beta(\bullet_{[p]}) = \beta(\bullet_{[q]})$ for every eBH $\beta : \mathbf{T}' \to \mathbf{A}$;

(ii) $\mathbf{A} \dot{\models}^{\aleph}_{\mathrm{eBA}} p \simeq q$ iff $\beta(\bullet_{[p]}) = \beta(\bullet_{[q]})$ for each admissible eBH $\beta : \mathbf{T}' \dot{\to} \mathbf{A}$;

(iii) $\mathbf{A} \in Mod^{\aleph}(\Gamma)$ iff $\mathbf{A} \models^{\aleph}_{\mathrm{eBA}} p \simeq q$ for every $p \simeq q \in \Gamma$;

(iv) $\mathbf{A} \in Adm^{\aleph}(\Gamma)$ iff $\mathbf{A} \dot{\models}^{\aleph}_{\mathrm{eBA}} p \simeq q$ for each $p \simeq q \in \Gamma$.

Note that $\models^{\aleph_0}_{\mathrm{eBA}}$ and $\models_{\mathrm{eBA}}$ coincide with each other so that $Adm(\Gamma) = Adm^{\aleph_0}(\Gamma)$ as well as $\models^{\aleph_0}_{\mathrm{eBA}}$ and $\models_{\mathrm{eBA}}$ coincide with each other so that $Mod(\Gamma) = Mod^{\aleph_0}(\Gamma)$.

Similarly, we have the equational logic $\vdash^{\aleph}_{\mathrm{eBA}}$ corresponding to $\models^{\aleph}_{\mathrm{eBA}}$. Also, $\vdash_{\mathrm{eBA}}$ and $\vdash^{\aleph_0}_{\mathrm{eBA}}$ coincide with each other.

Recall the proof of the admissible completeness of $\vdash_{\mathrm{eBA}}$, by carefully examining the proof, we understand that the same proof can be carried through for $\vdash^{\aleph}_{\mathrm{eBA}}$ so that $\vdash^{\aleph}_{\mathrm{eBA}}$ is sound and admissibly complete. That is,

Theorem 9.3.3. *For any cardinal* $\aleph \geqslant \aleph_0$, $\Gamma \models^{\aleph}_{\mathrm{eBA}} p \simeq q$ *iff* $\Gamma \vdash^{\aleph}_{\mathrm{eBA}} p \simeq q$.

Actually, there is no essential difference between $\vdash_{\mathrm{eBA}}$ and $\vdash^{\aleph}_{\mathrm{eBA}}$ except that the number of the ordinary variables available in $\vdash_{\mathrm{eBA}}$ is countably infinite $\aleph_0$ and the number of the ones available in $\vdash^{\aleph}_{\mathrm{eBA}}$ is $\aleph (> \aleph_0)$. In this sense, we regard both $\vdash_{\mathrm{eBA}}$ and $\vdash^{\aleph}_{\mathrm{eBA}}$ as identical. Also, it can be verified that

(1) $Mod(\Gamma) = Mod^{\aleph_0}(\Gamma) = Mod^{\aleph}(\Gamma)$ for each $\aleph \geqslant \aleph_0$;

(2) $Adm(\Gamma) = Adm^{\aleph_0}(\Gamma) \supseteq Adm^{\aleph}(\Gamma)$ for every $\aleph \geqslant \aleph_0$.

So, the result of Lemma 9.3.2 can be generalized. That is,

Lemma 9.3.4. *Let* $\mathbf{A} = \langle\langle A, \mathscr{F}\rangle, \mathscr{A}\rangle$ *be an eBA such that* $\aleph_0 \leqslant |A| \leqslant \aleph$, *then* $\mathbf{A} \models^{\aleph}_{\mathrm{eBA}} p \simeq q$ *iff* $\mathbf{A} \models^{\aleph}_{\mathrm{eBA}} p \simeq q$.

From this lemma, we have that $\mathbf{A} \in Adm^{\aleph}(\Gamma)$ and $\aleph_0 \leqslant |A| \leqslant \aleph$ implies $\mathbf{A} \in Mod(\Gamma)$. So, for $\aleph \geqslant \aleph_0$, if $\mathbf{A} \in Adm(\Gamma)$ and $|A| = \aleph$, then $\mathbf{A} \in Adm^{\aleph}(\Gamma)$ implies $\mathbf{A} \in Mod(\Gamma)$. Hence, a possible difference can only lie between $Adm(\Gamma)$ and $Adm^{\aleph}(\Gamma)$, i.e. whether there is an eBA $\mathbf{A} \in Adm(\Gamma)$ such that $\mathbf{A} \notin Adm^{\aleph}(\Gamma)$ for some $\aleph > \aleph_0$.

If there is no such an eBA, then $\vdash_{\mathrm{eBA}}$ is sound and complete, i.e. the completeness and the admissible completeness of $\vdash$

$_{\mathrm{eBA}}$ are one. Furthermore, the freeness requirement on term eBAs is too strong for the completeness and a weaker one like admissible freeness is enough. But we suspect that there exists such an eBA. However, we fail to construct an example to justify our suspicion. Basically, the reason for that is: if such an example (an eBA) exists, it must have uncountable ordinary objects (see Lemma 9.3.2); e.g.

- in its function spaces, it has two different functions which agree on countable ordinary objects which forms a part of a perfect sub-eBA;
- on the other hand, it may not have a perfect sub-eBA which has countably ordinary objects.

We leave these as open problems.

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